We’re here to help!

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We’re here to help you!
The Final Exam

- Look over everything you know and have ever done in this class – quizzes, sample problems, readings, etc.
- The syllabus is a good place to start if you’re trying to remember everything we’ve covered.
- Just because I did or did not put something in this PowerPoint doesn’t mean it is or is not fair game. Try as I may, I can’t read Dr. Melton’s mind!

- Make use of our office hours!
The Final Exam (Cont’d)

- Reminders:
  - We drop your lowest quiz score
  - We do not curve either the exams or the course

- Ask questions if you need help! It’s never too late* to ask a question about course material!

- Time and Location: Wednesday, May 7th, 10:10 – Noon in 22 BBH Building

- Exam covers everything in the course!

*Until we hand you the final exam
What is Dynamics?

- **Dynamics**: The exchange, dissipation and addition of total energy.
- **Can be stored in different forms**:  
  - Kinetic: Mass, Rotational  
  - Potential: Springs, Structures, Gravity Fields
- **Energy can also be dissipated**:  
  - Friction, drag, damping, etc.
- **Equations of Motion (EOMs)**: The differential equations that describe the motion of a body or a dynamical system.
- **Response**: The solution to the differential equations (EOMs). The number of EOMs is equal to the number of DOFs for any system!
A note about linearity

- Most (interesting) systems are non-linear
  - Why? Because the real world is complicated...
  - Non-linear problems are generally very difficult to solve analytically and the principle of superposition generally does not hold for non-linear systems.

- How do we overcome this?
  - Special Cases
  - Linearize the system
    - Binomial Expansions
    - Series Expansion (e.g. Power Series, Taylor Series, etc.)
  - Solve it numerically (solvers and methods include ODE45, Gauss-Jackson, Runge-Kutta, etc.)
Unit Impulse

By definition

\[ F(t; a) = \begin{cases} 
0, & t \leq \frac{-a}{2} \\
1, & \frac{-a}{2} < t < \frac{a}{2} \\
\frac{1}{a}, & \frac{a}{2} < t \leq \frac{a}{2} \\
0, & t \geq \frac{a}{2} 
\end{cases} \]

As a gets smaller, it approaches the Dirac Delta function

\[ \lim_{a \to 0} F(t; a) = \int \delta(t) \, dt \]
Dirac Delta

- Dirac Delta’s are impulse functions

\[ \delta(t) = \begin{cases} 
0, & \text{if } t \neq 0 \\
\infty, & \text{if } t = 0 
\end{cases} \]
Step function

Definition

\[ a \ u(t - t_o) = a \int_{-\infty}^{t} \delta(t - t_o) \, dt \]

\[ a \ u(t - t_o) = \begin{cases} 0, & \text{if } t < t_o \\ a, & \text{if } t > t_o \end{cases} \]

Like in the homework, we can use step functions to create ramp functions.

\[ \text{Step function} \]

\[ \text{on/off} \]
Some old AERSP 309 concepts

- Particle Dynamics

\[
\begin{align*}
\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\
\vec{v} &= \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \\
\vec{a} &= \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}
\end{align*}
\]

- Linear Momentum

\[
\vec{L} = m\vec{v}
\]

- Angular Momentum

\[
\vec{H} = I\vec{\omega}
\]

- Deformable Body

\[
\dot{\tau}_{B/A} \neq 0
\]
Vector Derivatives

- This slide was copied from my AERSP 309 final exam review (this is still important material, though!)

- Derivatives of vectors in other frames of reference
  - First Derivative
    \[
    \frac{d\vec{r}}{dt} = \frac{B\vec{r}}{dt} + \vec{\omega}_B^I \times \vec{r}
    \]
  - Second Derivative
    \[
    \frac{d^2\vec{r}}{dt^2} = \frac{B\vec{r}}{dt^2} + \frac{d\vec{\omega}_B^I}{dt} \times \vec{r} + 2\vec{\omega}_B^I \times \frac{B\vec{r}}{dt} + \vec{\omega}_B^I \times (\vec{\omega}_B^I \times \vec{r})
    \]

Strategy: Never compute anything twice!
Some EMCH Concepts

- Center of Mass (not to be confused with the center of gravity)
  \[ \overline{r} = \frac{1}{m} \int \overline{r} \, dm \]

- Moments of Inertia – Can be tabulated into Inertia Tensors (Matrices)
  \[ I_{AA} = \int r^2 \, dm \]

- Parallel Axis Theorem
  \[ I_{total} = \sum_{i=1}^{N} (I_{i,cm} + m_i r_i^2) \]
Degrees of Freedom (DOF)

- Particle
  \[ n_p = 3 - k \]

- Rigid Body
  \[ n_p = 6 - k \]

- Total DOF for a System
  \[ n_{p,\text{Total}} = \sum_{i=1}^{N} n_{p,i} \]

  k is the number of constraints
Examples

3-DoF
Some useful relationships include:

\[ F(t) = k\Delta x \]

\[ F(t) = c\Delta v \]

\[ F(t) = I \delta(t - t_o) \]

\[ F(t) = F_o u(t - t_o) \]

\[ M_A = -K_t \theta \]
Combinations

- **Springs**
  
  Parallel: \( k_{eq} = \sum_{i=1}^{N} k_i \)
  
  Series: \( k_{eq} = \frac{1}{\sum_{i=1}^{N} \frac{1}{k_i}} \)

- **Dampers**
  
  Parallel: \( c_{eq} = \sum_{i=1}^{N} c_i \)
  
  Series: \( c_{eq} = \frac{1}{\sum_{i=1}^{N} \frac{1}{c_i}} \)

- Beware of springs or dampers that look to be in series but act in parallel (and vice-versa)
D’Alembert’s Principle

D’Alembert’s Principle allows you to convert a dynamics problem into a statics problem. Statics problems are usually easier to solve.

\[ \sum \vec{F} = m \vec{a} \rightarrow \sum \vec{F} - m \vec{1} \vec{a} = 0 \]

\[ \sum \vec{M} = \frac{d\vec{H}}{dt} \rightarrow \sum \vec{M} - \vec{1} \frac{d\vec{H}}{dt} = 0 \]
Free Body Diagrams

- These should be familiar from high school physics, PHYS 211, E MCH 210 (or E MCH 211), E MCH 212, etc. so I won’t belabor the point too hard.
- Draw blocks to represent your figures
- You need some sort of sign convention
- Draw the directions in which the forces and/or moments are acting

\[ \sum F: m\ddot{x} = F_1 + F_2 + \ldots \]
\[ \sum M: I\ddot{\theta} = M_1 + M_2 + \ldots \]

Remember: Moments and forces are related by the equation \( M = Fd \) (where \( d \) is your moment arm)
Spring-Mass-Damper

- One of the most basic systems is the spring-mass-damper system

\[ \sum F_x: \quad m\ddot{x} = -kx - c\dot{x} + f(t) \]

Always watch your sign conventions!
Single vs. Multiple DOF Systems

- **SDOF System**
  \[ m\ddot{x} + c\dot{x} + kx = f(t) \]

- **MDOF System**
  \[ M\dddot{x} + C\ddot{x} + Kx = \vec{F}(t) \]

Be able to put MDOF Systems into matrix form! This shows up throughout the course (e.g. forming impedance matrices, etc.)
Lagrange’s EOMs

- Suppose we have an $n$-DOF system, then

$$T = \text{Kinetic Energy of the System}$$
$$V = \text{Potential Energy of the System}$$
$$Q_i = \text{generalized external force}$$

We can then form the Lagrangian

$$L = T - V$$
Lagrange’s EOMs Cont’d

For $i = 1, 2, \ldots, n$

Non-Conservative Systems:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = Q_i$$

Conservative Systems:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} = 0$$
What is $Q_i$?

- $Q_i$ contains the generalized forces or moments that were not included in the Lagrangian.

- Friction, forcing function, and any other velocity-dependent terms (including damping and drag) are included.
  - In other words, non-conservative effects are included.

- We calculate $Q_i$ using the principle of virtual work.
Principle of Virtual Work

- Imagine the system is in motion \((q_i, \dot{q}_i)\) are nonzero for \(i = 1, \ldots, n\).
- At some time \(t\), displace \(q_i\) by \(\delta q_i\)
- Then the virtual work done by the \(Q_i\) is

\[ \delta W = \sum_{i=1}^{n} Q_i \delta q_i \]

and

\[ Q_i = \frac{\delta W}{\delta q_i} \]

- Free body diagrams can help you figure out what is happening with the generalized forces.
Laplace Transforms

- These apply only to linear systems (i.e. the EOMs must be linear)
- Laplace transforms permit us to solve (relatively difficult to solve) differential equations by converting them into (relatively simple to solve) algebraic equations.
- This gives us a more mechanical (procedural) approach to these problems (as opposed to the more special case (“here’s how I approach this one particular kind of problem”) approach that many of you might have struggled with in MATH 250 or MATH 251)
Laplace Transforms Cont’d

- Laplace Transforms can often be combined with other techniques such as partial fraction expansions or partial fraction decompositions.

- Definitions:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt = F(s)$$

$$L^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) \, ds = f(t)$$

- In practice, we don’t use these definitions (some of you tried to brute force these on the first midterm) - use look-up tables instead! Sometimes you have to manipulate the expression to make it match up with what is in the table – this is generally not too bad to do.
The Beautiful Properties of the Laplace Transformation

- **Linearity (principle of superposition)**
  \[ \mathcal{L}\{a f_1(t) + b f_2(t)\} = a F_1(s) + b F_2(s) \]

- **Differentiation**
  \[ \mathcal{L}\left\{ \frac{df(t)}{dt} \right\} = s F(s) - f(0) \]
  \[ \mathcal{L}\left\{ \frac{d^2 f(t)}{dt^2} \right\} = s^2 F(s) - s f(0) - f'(0) \]

- **Integration**
  \[ \mathcal{L}\left\{ \int f(t)dt \right\} = \frac{F(s)}{s} \]

- **Final Value Theorem**
  \[ \lim_{t \to \infty} x(t) = \lim_{s \to 0} s F(s) \]
Partial Fraction Decomposition

\[ X(s) = R(s) \frac{N(s)}{D(s)} \]

- \( R(s) \) is a non-polynomial function of \( s \)
- \( N(s) \) is a numerator polynomial of order \( p \)
- \( D(s) \) is a denominator polynomial of order \( n \)

For a physical system, \( n \geq p \) but the case of \( p > n \) rarely happens. (If it does, Google™ it.)
Partial Fractions Cont’d

- There are several cases you should be aware of (and know how to deal with)
  - Distinct, real roots
  - Complex roots (will always appear in complex conjugate pairs)
    - Approach 1: Don’t factor it (solve for two unknowns)
    - Approach 2: Treat as distinct roots
    - Try using phasors! The math is much easier in many cases than trying to brute force the algebra.
  - Repeated real roots
  - Combinations of these (you saw something like this on the second midterm)
Transfer Functions

\[
G(s) = \frac{\mathcal{L}\{x(t)\}}{\mathcal{L}\{f(t)\}} = \frac{X(s)}{F(s)}
\]

More generally, we can form a matrix

\[
\tilde{X}(s) = K(s)\tilde{F}(s)
\]

Remember what we did to find the system response if we had the transfer function and knew what the input was.
A Couple of Strategies for Working with Transfer Functions

- Polar form of complex numbers – sometimes is easier to work with mathematically
- Partial fraction decomposition – makes nasty fractions into smaller, easier to manage ones
- Laplace table – if your s-domain result looks like something from the table, use the table!
- Take advantage of mathematical tricks such as complex conjugates to save precious time
- Watch any/all quadrant checks!
System Inputs

- There are several different commonly seen inputs to a system
  - Free Response: $F(t) = 0$
    - Typically with non-zero initial conditions.
    - Sometimes we’ll say “ignore initial conditions” – in this case, assume all necessary initial conditions are zero. You need to understand what’s going on with the transfer functions!
  - Step input: $F(t) = Au(t - t_o)$
  - Impulse input: $F(t) = B\delta(t - t_o)$
  - Ramp Response: $F(t) = C(t - t_{0,1})u(t - t_{0,2})$
Transient Response

Some key definitions to know – these are all important characteristics of a system’s response

- **Final Value**: The steady state or final value of the response of the system (can often be found via the final value theorem)
- **2% Settling Time**: Time it takes for the response to enter and stay within 2% of the final value
- **10-90% Rise Time**: Time it takes for the response to go from 10% to 90% of the final value
- **Percent Overshoot**: Percent of the max load of the input, calculated from the equation

\[ \eta_o = 100\% \frac{x_{peak} - G(0)}{G(0)} \]

- Know the approximation formulas from the notes
Stability Analysis

- Couple different techniques
  - Look at the poles of the system
  - Routh’s Table/Criterion
    - Number of sign changes is equal to the number of poles with positive real parts. We made heavy use of these!
    - Graphical methods such as Root Locus, Bode plots or Nyquist plots. We talked about the first two of these, but be aware there are others used in the real world.

- Stability Classifications
  - Asymptotic stability
    - Stable (decays to zero)
    - Neutrally Stable (steady state) – often rather sensitive to perturbations that can lead to instability
    - Unstable (blows up!)
  - Stability is $L$ (in the sense of Lyapunov)
  - Many more definitions of stability exist in the literature.
Stability and Relative Stability

- To make a system stable, you want all of the poles to be in the left half plane (i.e. have negative real parts)
- We can also say a system is stable relative to some (arbitrary) condition
Root Locus

- Transfer function of the form

\[ G(s) = \frac{X(s)}{F(s)} = \frac{N(s)}{D(s)} \]

- The denominator can be rewritten as

\[ D(s) = Q(s) + KR(s) = 0 \]

- So what’s going on?
  - \( K = 0 \rightarrow \) Poles are roots of \( Q \)
  - \( K = \infty \rightarrow \) Poles are roots of \( R \)
  - Look at the Root Locus plot – see where the plot crosses the imaginary axis; look at the gain value \( K \)
Second Order Systems

- **Standard form of a differential equation**

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = \frac{1}{m} f(t) \]

- **Transfer Function**

\[ G(s) = \frac{As + B}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

- **Motion Input**

\[ \ddot{x} + 2\zeta \omega_n \dot{x} + \omega_n^2 x = 2\zeta \omega_n \dot{y} + \omega_n^2 y \]

- **Poles of the transfer function**

\[ s_{1,2} = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]
Second Order Systems Cont’d

- Undamped free response

\[ x(t) = A \sin(\omega_n t + \phi) \]

- Underdamped free response

\[ x(t) = A_d e^{-\zeta \omega_n t} \sin(\omega_d t + \phi) \]

- Critically damped free response

\[ x(t) = e^{-\omega_n t} + (\dot{x}_o + \omega_n x_o) + e^{-\omega_n t} \]

- Overdamped free response

\[
\begin{align*}
    x(t) &= \frac{1}{2\sqrt{\zeta^2 - 1}} \left\{ \left [ x_o \left ( -\zeta + \sqrt{\zeta^2 - 1} \right ) - \frac{\dot{x}_o}{\omega_n} \right ] e^{s_1 t} \\
    &\quad + \left [ x_o \left ( \zeta + \sqrt{\zeta^2 - 1} \right ) + \frac{\dot{x}_o}{\omega_n} \right ] e^{s_2 t} \right \}
\end{align*}
\]
Higher Order Systems

- Recall that we write these equations as

\[ M \ddot{x} + C \dot{x} + Kx = \vec{F}(t) \]

- We can then form the impedance matrix

\[ Z(s) = Ms^2 + Cs + K \rightarrow Z(s)X(s) = F(s) \]

- Frequencies are found from the roots of the determinant equation
Systems with Time Delay

- Response at a time $t$ is affected by the system's response at a previous time $t - \tau$ for a fixed value of $\tau$
- Use a modified version of the second shifting theorem

$$\mathcal{L}\{x(t - \tau)\} = e^{-\tau s}x(s)$$
State Space and Numerical Integration

- It’s hard to test you on numerical integration in a class like this. Instead, focus on knowing how to put equations in State Space form.
- We use State Space form for numerical integration and control analysis.
- In essence, you convert higher-order linear or non-linear differential equations into first order differential equations (which in theory are easier to solve).
- No derivatives on the right hand side!
State Space Example

- Assume we have an equation of the form

\[ A\ddot{y} + B\dot{y} + C\dot{y} + Dy = f(t) \]

where \( A, B, C \) and \( D \) are constants \( \in \mathbb{R} \)

Dependent variable: \( y \)

Highest derivative: 3

\( 1 \text{ state } \times 3 \text{ derivatives} = 3 \text{ equations} \)
State Space

Let

\[ x_1 = y \]
\[ x_2 = \dot{x}_1 = \dot{y} \]
\[ x_3 = \dot{x}_2 = \ddot{y} \]

Substituting, we then find

\[ A\dot{x}_3 + Bx_3 + Cx_2 + Dx_1 = f(t) \]

Finally, we get

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= \frac{1}{A} [f(t) - Bx_3 - Cx_2 - Dx_1]
\end{aligned}
\]
Steady-State Frequency Response

- Sinusoidal input

\[ f(t) = F_o \sin(\omega t) \]

- We assume that our system is asymptotically stable, we can calculate

\[ G(iw) = G_R + iG_I \]

and

\[ |G(iw)| = \sqrt{G_R^2 + G_I^2} \]
We need a quadrant check for the phase

\[ \phi = \begin{cases} 
\tan^{-1} \left( \frac{G_I}{G_R} \right), & G_R \geq 0 \\
\tan^{-1} \left( \frac{G_I}{G_R} \right) + \pi, & G_R < 0 
\end{cases} \]

Putting it all together, we have

\[ x_s(t) = F_0 |G(i\omega)| \sin(\omega t + \phi) \]
Bode Plots

- A Bode plot is a plot of $20 \log |G|$ and $\phi$ vs. $\omega$
- In class we discussed a procedure for sketching asymptotic Bode plots (we’ll handle special cases separately)

1. Convert $G(s)$ from the form

$$G(s) = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + z_n)}$$

   to the form

$$G(s) = \frac{K'(1 + \tau'_1s)(1 + \tau'_2s) \cdots (1 + \tau'_m s)}{(1 + \tau_1s)(1 + \tau_2s) \cdots (1 + \tau_n s)}$$
Bode Plots Cont’d

Procedure Cont’d

2. Calculate the corner frequencies
3. Begin the Bode plots one decade below the lowest corner frequency. The starting amplitude is $20 \log |K'|$ with a starting slope of 0 dB/dec.
4. At each corner frequency, the slope of the amplitude will change +20 dB/dec if the corner frequency is in the numerator and -20 dB/dec if the corner frequency is in the denominator.
5. At each corner frequency, the magnitude of the phase will jump +90 deg if the corner frequency is in the numerator and -90 deg if the corner frequency is in the denominator.
6. Continue plotting until you are at least one decade above the highest corner frequency.
Bode Plots Cont’d

- **Procedure changes for special cases**
  - Separate factor of $s$ in the denominator of $G$

  \[
  G(s) = \frac{K'(1 + \tau_1's)(1 + \tau_2's)\cdots(1 + \tau_m's)}{s^p(1 + \tau_1's)(1 + \tau_2's)\cdots(1 + \tau_n's)}
  \]

  Then start the amplitude plot with initial value $20 \log \left(\frac{K'}{\omega^p_o}\right)$ and with an initial slope of $-20^\circ p$ dB/dec. The initial phase will be $90^\circ p$ deg.

- **Quadratic terms**
  - If quadratic terms appear, the slope of the amplitude changes by $\pm 40$ dB/dec and the phase will shift by $\pm 180$ deg.
Block Diagrams and Block Algebra

- Always reduce your answers as much as time allows!
- Steady state error calculations (error is a function of system type and input)

- Summing junction

- Negative summation
Example
Closed-loop transfer function

\[ p = G_1 r \]
\[ e = p - d \]
\[ c = G_2 e \]
\[ d = H c \]

Want \[ c = \frac{G_1 G_2}{1 + G_2 H} \]

\[ c = G_2 (p - d) = G_2 (G_1 r - H c) \]
\[ c = G_2 G_1 r - G_2 H c \]
\[ c (1 + G_2 H) = G_1 G_2 r \]
\[ c = \left[ \frac{G_1 G_2}{1 + G_2 H} \right] r \]

A open loop transfer function
Controller considerations

- P = proportional, D = derivative, I = integral
- Various Combinations: PI controller, PD controller, PID controller, PID controller with filtering for the derivative action, ...
- Tuning controllers is a bit of an inexact science (more on this in more advanced courses – for example, the Ziegler-Nichols tuning method)
- Routh tables are one approach, there are other methods you’ll learn in more advanced courses (Root Locus, other MATLAB tools, etc.)
That’s all, folks!

- Thanks for being a good class this year 😊