Higher Index Theory with change of fundamental group

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Dirac operators and index

We consider closed manifolds M and Dirac-type operators D on Clifford bundles S over them. The canonical ("Atiyah-Singer") Dirac operator on a spin manifold is an example, and for this discussion there is little loss of generality in restricting attention to it.

If M is *even* dimensional, the Clifford bundle S will be *graded* $S = S^+ \oplus S^-$, and in this case the *index* of D is the integer

$$\mathrm{Index}(D) = \dim(\ker(D) \cap C^\infty(S^+)) - \dim(\ker(D) \cap C^\infty(S^-)).$$

Observation: If D is *invertible* (for instance if M carries a metric of positive scalar curvature, in the classical case), then $\mathrm{Index}(D) = 0$.

Cobordism invariance

Theorem (Atiyah, Seeley, Singer)

Suppose that the closed spin manifold M^{2n} is the boundary of another compact spin manifold W^{2n+1}. Then $\mathrm{Index}(D_M) = 0$*.*

This could be said to follow from the Atiyah-Singer index theorem (since Pontrjagin numbers are necessarily cobordism invariants). An analytical demonstration of this result was however a critical step in the original *proof* of the index theorem.

We will describe two more modern approaches to a proof.

Approach I: K-homology

Use the facts that a (graded) elliptic operator D on a closed M corresponds to a K-homology class $[D] \in K_0(M)$, and that taking the index is the induced map on K-homology coming from $M \to \text{pt}$.

Let $M = \partial W$ and consider the diagram

$$\begin{array}{ccccc}
K_1(W, \partial W) & \xrightarrow{\partial} & K_0(\partial W) & \longrightarrow & K_0(W) \\
 & & \Big\downarrow \text{Index} & & \Big\downarrow \\
 & & \mathbb{Z} = K_0(\text{pt}) & \xrightarrow{\cong} & K_0(\text{pt})
\end{array}$$

where the first row is part of the exact K-homology sequence of the pair $(W, \partial W)$.

Lemma

The Dirac operator on the open manifold W° gives rise to a class in $K_1(W, \partial W)$, which maps to $[D_{\partial W}]$ under the boundary map. ("The boundary of Dirac is Dirac".)

The cobordism invariance of the index follows from this and simple diagram-chasing, using the fact that we have an isomorphism in the bottom row.

Approach II: coarse geometry

Suppose that V is a noncompact (complete Riemannian) *partitioned manifold* of dimension $2n + 1$, that is, $V = V^- \cup V^+$ with $V^- \cap V^+$ a compact hypersurface M.

One can define $\mathrm{Index}(D_V) \in K_1(C^*(V))$ and the partition P of V gives rise to a homomorphism $\varphi_P \colon K_1(C^*(V)) \to \mathbb{Z}$.

Theorem (Partitioned manifold index theorem)

In the above situation $\varphi_P(\mathrm{Index}\, D_V) = \mathrm{Index}(D_M)$.

Since φ_P does not change if P is compactly perturbed, the cobordism invariance of $\mathrm{Index}(D_M)$ follows.

The higher index

Let M be a closed manifold, as before, and D an elliptic operator. Consider a *flat* vector bundle E over M (associated to a unitary representation of the fundamental group $\pi = \pi_1(M)$). Then one can *twist* D by E, obtaining a new elliptic operator D_E which has an index of its own.

The universal example is to consider the flat bundle whose fibers are $C^*(\pi)$. In this case the 'kernel' and 'cokernel' are modules over $C^*(\pi)$ and the *higher index* thus becomes an element of $K(C^*(\pi))$.

These ideas are formalized by the assembly map

$$A\colon K_*(M) \to K_*(C^*(\pi_1(M))).$$

Example: the torus

Suppose that $M = \mathbb{T}^n$, the n-torus. Then $\pi_1(M) = \mathbb{Z}^n$ and $C^*(\pi_1(M)) = C(\widehat{\mathbb{T}}^n)$, where $\widehat{\mathbb{T}}^n$ is a "dual" n-torus *(Mukai duality)*.

In this case the assembly map

$$A\colon K_*(\mathbb{T}^n) \to K^*(\widehat{\mathbb{T}}^n)$$

can be understood using the index theorem for families. It is an isomorphism.

In particular $A[D] \neq 0$ for the Dirac operator, even though $\mathbb{T}^n$ is a boundary. Thus *the cobordism invariance of the ordinary index apparently does not extend to the higher index.* What went wrong?

The $\pi - \pi$ theorem

The homology argument (or the coarse geometry argument) for cobordism invariance will work *provided that the inclusion $\partial W \to W$ induces an isomorphism on fundamental groups*.

$$\begin{array}{ccccc} K_1(W, \partial W) & \xrightarrow{\partial} & K_0(\partial W) & \longrightarrow & K_0(W) \\ & & \downarrow A_{\partial W} & & \downarrow A_W \\ & & K_0(C^*(\pi_1(\partial W))) & \xrightarrow{\cong} & K_0(C^*(\pi_1(W))) \end{array}$$

This situation is reminiscent of the *$\pi - \pi$ theorem* in Wall's surgery theory.

Relative invariant

In general we should expect a *relative* index diagram

$$\begin{array}{ccccc} K_1(W, \partial W) & \xrightarrow{\partial} & K_0(\partial W) & \longrightarrow & K_0(W) \\ \downarrow A_{\mathrm{rel}} & & \downarrow A_{\partial W} & & \downarrow A_W \\ K_1(C^*(\pi_1(W), \pi_1(\partial W))) & \longrightarrow & K_0(C^*(\pi_1(\partial W))) & \longrightarrow & K_0(C^*(\pi_1(W))) \end{array}$$

that accounts for the failure of isomorphism in the bottom row.

Remark: To have functoriality under non-injective group homomorphisms it is necessary to use the *maximal* group C^*-algebras in the bottom row here.

Positive scalar curvature

To see why such an invariant might be of interest, consider the relationship of the Dirac operator to positive scalar curvature.

- If closed M has pscm then $A[D_M] = 0$.
- Suppose $(W, \partial W)$ has fixed pscm on boundary. This data defines $[D_W] \in K_*(W)$. We have $A[D_W] = 0$ if the boundary pscm extends to a collared pscm on the interior.
- Expected theorem: $[D_W] \in K_*(W, \partial W)$ defined *without* boundary conditions, and $A_{\mathrm{rel}}[D_W] = 0$ if any collared pscm on W.

Project

Chang, Weinberger and Yu use this ‘expected theorem’ to construct a non-compact manifold without a uniformly pscm that admits an exhaustion by compact manifolds with pscm. The example can even be taken to be contractible.

1. Give a coarse-geometric construction of $C^*(\alpha)$, where $\alpha\colon \pi_1(\partial W) \to \pi_1(W)$ is a group homomorphism, and of the relative assembly map A_{rel}.
2. Prove the ‘expected theorem’ above.
3. Relate to the analytic surgery sequence.

Ideas

1. Zeidler perspective on the assembly map. Zeidler, Rudolf, "Positive Scalar Curvature and Product Formulas for Secondary Index Invariants.", to appear in *Journal of Topology* arXiv:1412.0685, `http://arxiv.org/abs/1412.0685`.
2. Coarse-geometric approach to relative index theory. Roe, John. 1991. "A Note on the Relative Index Theorem." The Quarterly Journal of Mathematics. Oxford. Second Series 42(167):365–373.

The Zeidler approach to assembly

- Represent K-theory in the spectral picture.
- Represent K-homology by K-theory of the localization algebra. Can take coefficients in a flat bundle of Hilbert modules.
- An element of the localization algebra is a family of operators parameterized by $t > 0$. Assembly is the K-theory map induced by evaluation at some fixed t-value, e.g. $t = 1$.
- Both index and homology class of Dirac operator are represented by the functional calculus.

Relative theory

We consider W with boundary $M = \partial W$. Form W_∞ by adding a tube $M \times \mathbb{R}^+$ to ∂W and $M_\infty = M \times \mathbb{R}$. Let E_W and E_M be the canonical flat $C^*(\pi_1(W))$ and $C^*(\pi_1(M))$ bundles over these things. Form the coarse algebras $C^*(W_\infty; E_W)$ and $C^*(M_\infty; E_M)$. Let I_W and I_M be the left-hand ($x \leqslant 0$) ideals in these two algebras. Then there is a canonical homomorphism

$$\alpha\colon C^*(M_\infty; E_M)/I_M \to C^*(W_\infty; E_W)/I_w,$$

coming from the coarse equivalence of the right-hand tails of the two spaces together with the homomorphism $C^*(\pi_1(M = \partial W)) \to C^*(\pi_1(W))$ induced by $\partial W \to W$.

Let D be the double of $C^*(W_\infty; E_W)$ and $C^*(M_\infty; E_M)$ along the homomorphism α.

Relative theory (continued)

The above construction can be carried out in exactly the same way on the level of localization algebras. Thus we obtain *two* algebras D, let's call them D (the original one) and D^{loc}, and a homomorphism (evaluation at $t = 1$) from D^{loc} to D which should represent some kind of assembly map.

We will prove

- The K-theory of the algebra D^{loc} is the K-homology of the pair $(W, \partial W)$;
- The K-theory of the algebra D is the relative group $K(C^*(\alpha))$, where $\alpha\colon \pi_1(\partial W) \to \partial(W)$.

Key diagram for both proofs

Let $A = C^*(W_\infty; E_W)$, $B = C^*(M_\infty; E_M)$, $C = A/I_W$.

We have a diagram (both ‘asymptotic’ and ‘non asymptotic’ versions)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_W & \longrightarrow & D & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_W & \longrightarrow & A & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the vertical maps are induced by change of rings ($\otimes C^*(\pi_1(W))$).

- Non asymptotic case: A has trivial K-theory by Eilenberg swindle, B and C are effectively deloopings of group C^*-algebras.
- Asymptotic case: vert maps are isomorphisms on K-theory, bottom row is Barratt-Puppe sequence.

Conclusions

- We can construct the relative assembly map using coarse geometry plus Zeidler's techniques;
- The main vanishing theorem ("expected theorem") is a simple consequence of the construction;
- Remains to do: discussion of relative analytic structure set, etc.
- Hopefully can lead to a more accessible approach to some geometric examples.