A portfolio of counter-examples

With answers

Consider each of the following claims. All of them are false, and most are based on common misconceptions. Devise a counter-example to show the claim is false.

Productive practice: Take a moment to “unpack” the terminology to list the requisite properties. Then make a plan in words for what a counterexample would look like (e.g., Translating “All parallelograms are rhombuses” into “I need a shape that has parallel sides but not all the sides have the same length.”)

Making the most of it: Once you have one counter-example, challenge yourself to make it as convincing as possible. This might involving finding a new example or it might be a way of presenting the example effectively (perhaps nudging the student toward finding it on his or her own).

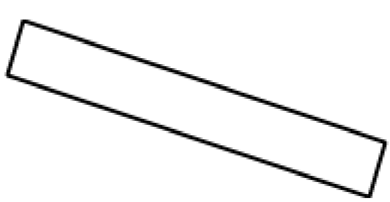
Sometimes, Always, Never: Most of these are misconceptions only because of the word “always” (either implied or explicit) instead of “sometimes” or even “often.” Perhaps it really is true of all examples the speaker had seen thus far, or at least they have been a victim of confirmation bias and ignored those examples that don’t fit expectations.

Answers have been added in italics. For many of these only one counter-example has been given but there are likely more. Commentary usually includes key features of a counter-example.

1. “A rectangle has two long sides and two short sides.”

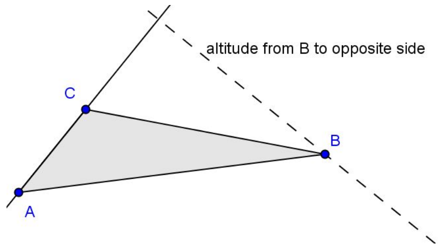
A square doesn’t have two “long” sides and two “short” sides, but a square does have four right angles and is therefore a rectangle. This misconception stems from a too-narrow working definition of rectangle.

2. “A rectangle has two horizontal sides and two vertical sides.”
   A “slanted” rectangle will work. This comes from seeing only rectangles with horizontal and vertical sides, which perhaps comes from saving space on the printed page.



3. “A triangle’s altitudes have to all be inside the triangle.” (The altitude is a line segment from a vertex to the opposite side, perpendicular to that side. It is the “height” of the area formula.)

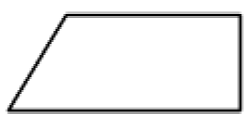
   Any obtuse triangle will have two altitudes lying outside the triangle. A right triangle has two altitudes on the sides of the triangle, which may or may not fall in the interior of the triangle depending on your definition.

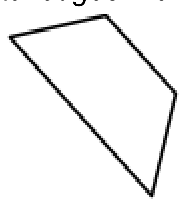


   Notice that the altitude does <u>not</u> need to run vertically. It needs to run <u>perpendicular</u> to the opposite side. These two ideas coincide only when the opposite side is horizontal.

4. “A trapezoid has two horizontal edges and two slanted sides.”

   Neither the “horizontal edges” nor “two slanted sides” are strictly necessary.


only one “slanted” side


no horizontal sides

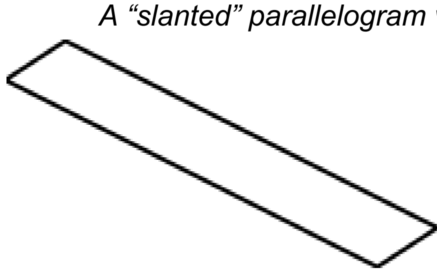
5. “The bases of a trapezoid are always different lengths.”

   If a quadrilateral has two parallel sides of the same length, then the other pair of sides will necessarily be parallel and so it must be a parallelogram.

   This means this statement may be true or false, depending on what you mean by “trapezoid.” If you define “trapezoid” as “a quadrilateral with <u>at least</u> one pair of parallel sides,” then parallelograms are also trapezoids and a parallelogram’s bases (parallel sides) are equal lengths. If you define as “a quadrilateral with <u>exactly</u> one pair of parallel sides,” then the statement is actually true.

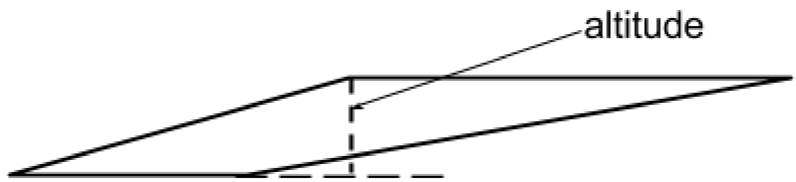
6. “A parallelogram always has two horizontal edges and two slanted sides.”

   A “slanted” parallelogram will work.



7. “All parallelograms are rhombuses.”
   Draw a parallelogram so that there are two different lengths of sides, like the one in the previous example.

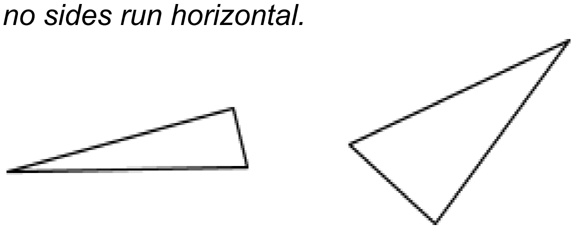
8. “The altitude of a trapezoid is always inside the trapezoid.” (The altitude is a line segment perpendicular to both bases running from one base to the other. It is the “height” of the area formula.)
   It is easy to forget how weird trapezoids can get. An altitude is a line segment perpendicular to both bases, but the segments of the bases might not be “long” enough.



9. “Any pentagon is a regular pentagon.”
   Any non-regular pentagon will suffice.

10. “There are four types of triangles: scalene, isosceles, equilateral, and right.” (Note: there is a lot going on here.)
    This student does not recognize the overlap between types, i.e., that there are angle-size-based categories and side-congruence-based categories. Several examples showing these combinations may be in order, e.g., scalene right triangle vs isosceles right triangle, acute isosceles vs obtuse isosceles.

11. “A right triangle always has a horizontal and a vertical side.”
    A “slanted” example is in order. The hypotenuse could run horizontal, or maybe no sides run horizontal.



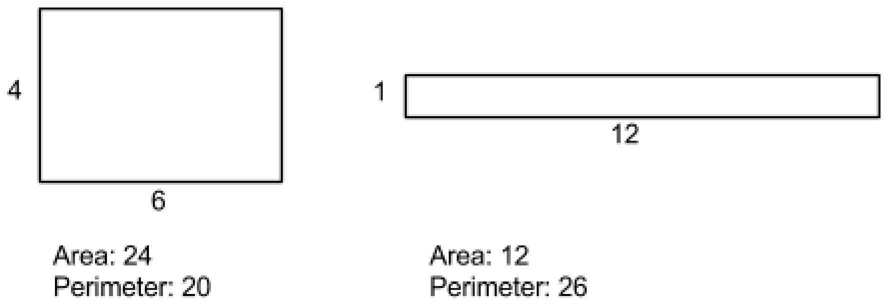
12. “The ‘height’ of a triangle (in an area calculation) is the length of any vertical side.”
    See the example in 3. Side AC is the base and the altitude marked is the corresponding height. Note that the height is not any side of the triangle.

13. “The ‘base’ of a triangle (in an area calculation) is the length of the bottom side.”
    See the example in 3. Side AC is the base, but is an “upper” side.

A note about base and height: *For the triangle area formula, the base and height work as a pair. There is no one side that is the base. Any of the three sides can be a base. However, once one side is settled on then the height must be the altitude dropped from the opposite vertex to that side. This means there are three options for any triangle for which is the base and which is the height. It is an enlightening exercise to draw a triangle, use a ruler to measure all base-height pairs, and find the area works out the same regardless of your choice.*

14. "This shape has larger area than that shape, so it has a larger perimeter too.". (Keep it simple: this can be done with rectangles.)

    This can be done with a pair of rectangles, A and B, where A has a larger area than B but a smaller perimeter.



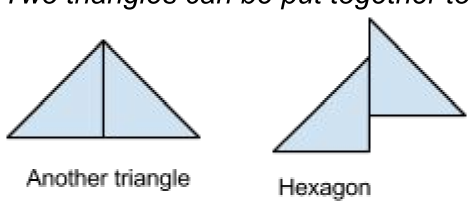
In general, as a rectangle gets closer to a square, its area grows and perimeter shrinks. Inversely, as a rectangle stretches into a longer shape its area shrinks and perimeter grows.

15. "This shape has larger perimeter than that shape, so it has a larger area too." (Keep it simple: this can be done with rectangles.)

    See 14 above.

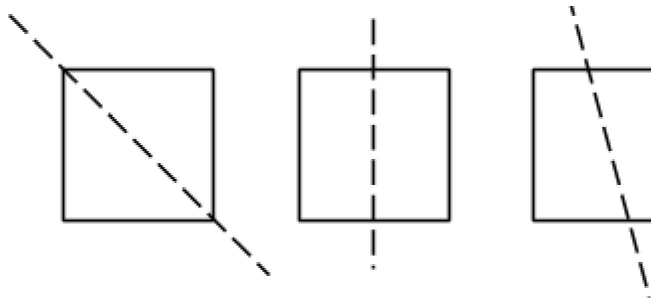
16. "Two triangles put together always make a rectangle."

    Two triangles can be put together to make all sorts of shapes.



17. "A square cut in half always makes two triangles."

    Cutting a square through the center can make triangles, rectangles, or *trapezoids.*



18. "A square cut in half always makes either two triangles or two rectangles."
    See 17 above.

19. "If a triangle has a 3-inch side and a 4-inch side, then the other side must be 5 inches long." (This only appears once student learn the Pythagorean Theorem.)
    One can easily draw an isosceles triangle with two 3-inch sides and a 4-inch side. One can even draw a right triangle with side-lengths 3 and 4 but the third side is not 5: make the hypotenuse length 4.

20. "That pentagon is regular because all the sides have the same length."
    It is challenging to free-hand, but you can draw a equilateral pentagon that is not equiangular (and thus not regular). The easiest way is to 5 rods of equal length (e.g., pencils or rulers or actual Cuisinaire rods) and arrange them on a table in a pentagon that is not a regular pentagon. Below are two more examples. Note that you can even have a non-convex equilateral pentagon.