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Benefit functions and duality

David G. Luenberger

Stanford University, Stanford CA, USA

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This paper studies a new representation of individual preferences termed the benefit function. The benefit function $b(g;\, x, u)$ measures the amount that an individual is willing to trade, in terms of a specific reference commodity bundle g, for the opportunity to move from utility level u to a consumption bundle x. The benefit function is therefore a generalization of the willingness-to-pay concept. This paper studies properties of this function, including its continuity and structural properties and its indirect relation to the underlying utility function. A very important property of the benefit function is that it is the natural precursor of the expenditure function, in the sense that the expenditure function is a (special) dual of the benefit function. This duality is shown to be complete by proving that when appropriate convexity properties hold, the (correspondingly special) dual of the expenditure function is, in fact, the benefit function. The duality makes the benefit function a powerful tool for analysis of welfare issues.

1. Introduction

The concept of individual preferences forms the foundation for the modern microeconomic theory of choice. The preference construct is rich enough to provide a general explanation for how individuals make economic decisions; and yet by purposely avoiding the assignment of absolute measures, it does not invite interpersonal comparisons. Indeed, it supports the common premise that such comparisons are meaningless.

Nevertheless, it is frequently convenient (and sometimes necessary) to, in fact, make interpersonal comparisons; such as when (certain types of) decisions are made that affect the welfare of a group of individuals. Such comparisons can be conducted formally by defining an overall societal preference relation (or welfare function). But a more direct technique is to represent individual preferences in a manner that facilitates combination. This approach has led to the concepts of willingness to pay, consumer surplus, and compensating and equivalent variations – which all have units of money, and therefore when any one of these is compared among individuals, they are all at least in comparable units. And, under various

Correspondence to: Prof. D.G. Luenberger, Department of Engineering–Economic Systems, Stanford University, Stanford, CA 94305-4025, USA.

special circumstances, it is actually meaningful to combine these measures across individuals.

This paper discusses another special measure – termed the *benefit function* – derived from individual preferences, which can be meaningfully combined across individuals. The benefit function measures utility with respect to a given reference utility level and in terms of the willingness to trade for a commodity bundle g. It therefore measures the benefit, in terms of g, of a move from a given utility level to a new bundle. This simple measure has a number of very remarkable and important properties. First, the benefit function is a natural precursor of the standard expenditure function; that is, the expenditure function is the dual of this function. In view of the importance of the expenditure function in microeconomic theory, the precursor status of the benefit function would seem, itself, to indicate that the benefit function is likely also to be very useful.

Second, and perhaps most importantly, the benefit functions of several individuals can be directly summed to obtain a meaningful aggregate benefit. This can be used like other aggregate measures to assess the welfare implications for changes in the economy.

This paper focuses on two main areas. The first is the general definition and properties of the benefit function. These include the continuity, concavity, algebraic, and monotonic properties. The second area is the duality between the benefit function to the expenditure function. It is shown that (in a certain sense) the dual of the benefit function is, in fact, the expenditure function. Then, to complete the duality, it is shown that the corresponding dual of the expenditure function is, under suitable convexity assumptions, equal to the original benefit function. This shows that benefit functions and expenditure functions form a natural dual combination.

2. The benefit function

Consider a world with d homogeneous and infinitely divisible commodities. A *bundle* of these commodities is a vector $x \in R^d$. Consider also a single consumer. This consumer has a *consumption possibility set* $\mathscr{X} \subset R^d$ from which bundles could be chosen if there were no economic constraints. For much of our work below, we assume that $\mathscr{X} = R^d_+$ (that is, the set of nonnegative vectors in R^d), but more general $\mathscr{X}$'s may also be considered. However, we shall always assume that $\mathscr{X}$ is closed, convex, and has a lower bound.

The consumer also has a preference ordering on $\mathscr{X}$ that can be represented by a continuous utility function u. We denote the range of u over $\mathscr{X}$ by $\mathscr{U}$. With these preliminaries, we introduce our main definition.

Definition. For any g, x, u with $g \neq 0$, $g \in R^d_+$, $x \in \mathscr{X}$, $u \in \mathscr{U}$ let

$$b(g;\, x, u) = \begin{cases} \max\{\beta : x - \beta g \in \mathscr{X}, u(x - \beta g) \geqq u\} \\ \qquad \text{if } x - \beta g \in \mathscr{X}, \text{ and } u(x - \beta g) \geqq u \text{ for some } \beta \\ -\infty \quad \text{otherwise.} \end{cases}$$

The function b defined this way is the *benefit function* associated with the utility function u and the consumption possibility set $\mathscr{X}$.

The maximization operation in the definition is well defined because u is continuous and $\mathscr{X}$ has a lower bound. There is a simple economic interpretation of $b(g;\, x, u)$. Basically, $b(g;\, x, u)$ is the amount of commodity bundle g that the consumer would be willing to trade for the possibility of moving from utility level u to the bundle x.

The vector g is a reference vector defining the measure by which alternative bundles are compared. It might be taken as $g = (1, 1, \ldots, 1)$ or as a specific commodity. Typically, the bundle g does not vary in a discussion.

There are some associated definitions which characterize various possible choices of the reference vector g.

Definition. The vector g is said to be *good* (that is, g is a good bundle) if for any $x \in \mathscr{X}$ there holds $x + \alpha g \in \mathscr{X}$ and $u(x + \alpha g) > u(x)$ for all $\alpha > 0$. The bundle g is *weakly good* if $x + \alpha g \in \mathscr{X}$ and $u(x + \alpha g) \geqq u(x)$ for all $\alpha > 0$. The bundle g is *locally good* or *locally weakly good* at $x \in \mathscr{X}$ if the above, respective, condition holds for all $\alpha \in (0, \bar{\alpha}]$ for some $\bar{\alpha} > 0$.

An important special case arises when a monotonicity assumption holds.

Definition. The utility function u is said to be *monotonic* if $\mathscr{X} = R^d_+$ and for any $x', x \in \mathscr{X}$ with $x' \geqq x$ there holds $u(x') \geqq u(x)$.

Note that we consider monotonicity to be an assumption about *both* $\mathscr{X}$ and u. Clearly, if u is monotonic, then any $g \neq 0$, $g \geqq 0$ is weakly good.

2.1. Fundamental properties

The remainder of this section examines the fundamental mathematical properties of the benefit function. In particular, we examine when the utility function can be recovered from the benefit function, and we shall explore the structural, continuity, concavity, and monotonicity properties. Finally we shall prove a converse theorem showing that any function with appropriate properties is, in fact, a benefit function of some utility function.

The utility function u can usually be recovered from the benefit function by solving the implicit equation

$$b(g;\ x, u)=0$$

for u in terms of x; but some special conditions are required for this to work – essentially the ray through x in the direction g (both positive and negative) must cut the set where utility has value u in at most one point. Special cases are spelled out in the following proposition.

Proposition 1. *(a) If g is good, then $u(x)=u$ implies $b(g;\ x,u)=0$.*
(b) If x is in the interior of $\mathscr{X}$, then $b(g;\ x,u)=0$ implies $u(x)=u$.

Proof. (a) Suppose that $u(x)=u$. Then clearly $b(g;\ x u)\geqq 0$. However, since g is good, $u(x-\beta g)<u(x)=u$ for any $\beta>0$. Hence, $b(g;\ x,u)=0$.
(b) If x is an interior point of $\mathscr{X}$, $b(g;\ x,u)=0$ implies that $u(x)\geqq u$ and $u(x-\beta g)<u$ for $\beta>0$. By continuity $u(x)=u$. □

The benefit function enjoys certain structural properties.

Proposition 2. The benefit function satisfies:
(a) Monotonicity: $b(g;\ x,u)$ is nonincreasing with respect to u.
(b) Translation: If $x\in\mathscr{X}$, $x+\alpha g\in\mathscr{X}$, then $b(g;\ x+\alpha g,u)=\alpha+b(g;\ x,u)$.
(c) Sign preservation: $u(x)\geqq u$ implies $b(g;\ x,u)\geqq 0$.
(d) Reverse sign preservation: If g is weakly good, then $b(g;\ x,u)\geqq 0$ implies $u(x)\geqq u$.

Proof. The first three of these follow immediately from the definition. To prove (d) suppose g is weakly good and $b(g;\ x,u)\geqq 0$. For $\beta=b(g;\ x,u)$ we have $u(x-\beta g)\geqq u$ with $\beta\geqq 0$. Since g is weakly good, $u(x)\geqq u(x-\beta g)\geqq u$. □

We turn next to the continuity properties of the benefit function. Normally, g is fixed in any development, so we are mainly interested in continuity with respect to x and u. However, we can consider continuity with respect to g as well. For generality, we shall temporarily relax our running assumption that u is continuous.

Proposition 3. Suppose u is upper semi-continuous. Then b is upper semi-continuous with respect to g, x, and u (jointly).

Proof. Fix $\bar{g}$, $\bar{x}$, $\bar{u}$ with $\bar{g}\neq 0$, and let $\bar{b}=b(\bar{g};\bar{x},\bar{u})$. We must show that given

$\varepsilon>0$ there is a neighborhood of $(\bar{g},\bar{x},\bar{u})$ such that for all (g,x,u) in this neighborhood

$$b(g;\ x,u)<\bar{b}+\varepsilon.$$

Suppose the converse. Then there is a sequence $\{(g_i,x_i,u_i)\}$ with $g_i\to\bar{g}$, $x_i\to\bar{x}$, $u_i\to\bar{u}$ and $b_i\equiv b(g_i;\ x_i,u_i)\geqq\bar{b}+\varepsilon$ for all i. We can assume that the g_i's are bounded away from 0. The sequence $\{b_i\}$ is bounded above because the range of β's such that $x_i-\beta g_i$ is feasible is bounded above (recall $\bar{g}\geqq 0$, $\mathscr{X}=R^m_+$). Hence, there is a convergent subsequence of $\{b_i\}$, which without loss of generality, we can take to be $\{b_i\}$ itself. Thus $b_i\to b_0$ for some b_0.

We have $u(x_i-b_ig_i)\geqq u_i$. By upper semi-continuity of u, it follows that $u(\bar{x}-b_0\bar{g})\geqq\lim u(x_i-b_ig_i)\geqq\bar{u}$. Hence $\bar{b}\geqq b_0$. But this contradicts $b_i\geqq\bar{b}+\varepsilon$ for each i. Therefore such a sequence cannot exist, and b is upper semi-continuous. □

If g is good, then stronger continuity results hold.

Proposition 4. Suppose that g is good and u is continuous. Then $b(g;\ x,u)$ is continuous with respect to x and u (jointly) in the interior of the region of $\mathscr{X}\times\mathscr{U}$ where $b(g;\ x,u)$ is finite.

Proof. In view of Proposition 3, we must only show that b is lower semi-continuous. Fix $\bar{x}\in\mathscr{X}$ and $\bar{u}\in\mathscr{U}$. Let $\bar{b}=b(g;\ \bar{x},\bar{u})$. Suppose b were not lower semi-continuous at $(\bar{x},\bar{u})$. Then there is $\varepsilon>0$ and a sequence $\{x_i,u_i\}$ such that $x_i\to\bar{x}$, $u_i\to\bar{u}$ and $b_i\equiv b(g;\ x_i,u_i)<\bar{b}-\varepsilon$ for all i. We know that $\bar{x}-\bar{b}g\in\mathscr{X}$. Hence, there is an I such that $x_i-(\bar{b}-\varepsilon/3)g\in\mathscr{X}$ for all $i>I$. We also have $u(x_i-b_ig)\geqq u_i$, and since b_i is the maximum value for which this inequality holds, it follows that $u(x_i-(b_i+\varepsilon/3)g)<u_i$.

Now let Ω be the closure of a neighborhood of $\bar{x}$ (relative to $\mathscr{X}$) such that for all $x\in\Omega$, $x-(\bar{b}-\varepsilon/3)g$ is feasible. Let

$$\delta=\min_{x\in\Omega}\{u(x-(\bar{b}-2\varepsilon/3)g)-u(x-(\bar{b}-\varepsilon/3)g)\}.$$

We have $\delta>0$ because g is good and u is continuous on Ω.

From $u(x_i-(b_i+\varepsilon/3)g)<u_i$ and $b_i<\bar{b}-\varepsilon$, we have $u(x_i-(\bar{b}-2\varepsilon/3)g)<u_i$. From the definition of δ,

$$u(x_i-(\bar{b}-\varepsilon/3)g)\leqq u(x_i-(\bar{b}-2\varepsilon/3)g)-\delta\leqq u_i-\delta.$$

Taking the limit, we obtain $u(\bar{x}-(\bar{b}-\varepsilon/3)g)\leqq\bar{u}-\delta$ which contradicts the definition of $\bar{b}$. Thus no such $\{(x_i, u_i)\}$ sequence can exist. Hence the function b is lower semi-continuous. □

One of the most important properties of the benefit function is that it is concave if u is quasi-concave.

Proposition 5. If the utility function u is quasi-concave, then $b(g;\, x, u)$ is concave with respect to x and continuous with respect to x on the interior of the region where it is finite. Conversely, if $b(g;\, x, u)$ is concave in x for all $u \in \mathscr{U}$, and g is weakly good, then the utility function is quasi-concave.

Proof. Assume u is quasi-concave. Let $x_1, x_2 \in \mathscr{X}$ be given. Select $u \in \mathscr{U}$. Suppose first that $b(g;\, x_1, u)$ and $b(g;\, x_2, u)$ are finite. Then, by definition,

$$u(x_1 - b(g;\, x_1, u)g) \geqq u, \qquad u(x_2 - b(g;\, x_2, u)g) \geqq u.$$

By quasi-concavity of the utility function

$$u(\alpha x_1 - \alpha b(g;\, x_1, u)g + (1-\alpha)x_2 - (1-\alpha)b(g;\, x_2, u)g) \geqq u,$$

for any α, $0 \leqq \alpha \leqq 1$. This means that

$$b(\alpha x_1 + (1-\alpha)x_2) \geqq \alpha b(g;\, x_1, u) + (1-\alpha)b(g;\, x_2, u).$$

If either $b(g;\, x_1, u) = -\infty$ or $b(g;\, x_2, u) = -\infty$, it is true by default that $b(g;\, \alpha x_1 + (1-\alpha)x_2) \geqq -\infty$ and hence the concavity relation holds. The continuity property stated in the proposition is true for any concave function.

Conversely, suppose b is concave. Suppose x_1 and x_2 satisfy $u(x_1) \geqq u$, $u(x_2) \geqq u$. By Proposition 2(c), $b(g;\, x_1, u) \geqq 0$, $b(g;\, x_2, u) \geqq 0$. By concavity,

$$b(g;\, \alpha x_1 + (1-\alpha)x_2, u) \geqq 0.$$

Then if g is weakly good, Proposition 2(d) gives $u(\alpha x_1 + (1-\alpha)x_2) \geqq u$. Hence the function u is quasi-concave. □

We show that the benefit function inherits monotonicity that might be possessed by the underlying utility function.

Proposition 6. Let $\mathscr{X} = R^d_+$.
(a) If u is monotonic on $\mathscr{X}$, then $b(g;\, x, u)$ is monotonic with respect to x.
(b) If u is strongly monotonic and continuous on $\mathscr{X}$, then in the interior of the

region where $b(g;\, x,u)$ is finite, $b(g;\, x,u)$ is continuous and strongly monotonic with respect to x.

Proof. (a) Fix $u\in\mathscr{U}$ and suppose $x'\geqq x$, with $x\in\mathscr{X}$, $x'\in\mathscr{X}$. Let $b=b(g;\, x,u)$ and assume $b>-\infty$. Then $u(x-bg)\geqq u$. By monotonicity $u(x'-bg)\geqq u$. Hence, $b(g;\, x',u)\geqq b$. If $b=-\infty$, it is clear again, by default, that $b(g;\, x',u)\geqq b$.
(b) Fix $u\in\mathscr{U}$ and suppose $x'\geqq x, x'\neq x$, with $x\in\mathscr{X}$, $x'\in\mathscr{X}$. Let $b=b(g;\, x,u)>-\infty$. Then $u(x-bg)\geqq u$. By strong monotonicity $u(x'-bg)>u$. Continuity of the utility function then implies $b(g;\, x',u)>b$. Continuity of $b(g;\, x,u)$ with respect to x follows from Proposition 4 since under strong monotonicity any $g\geqq 0, g\neq 0$, is good. □

Finally, we give the following converse proposition which shows which properties of a function insure that it is in fact a benefit function. Aside from continuity, the properties required are the translation property and the property that $b(g;\, x,u)$ be nonincreasing in u.

Proposition 7. Let $\mathscr{U}$ be a closed interval on the real line and let b be a function defined on $\mathscr{X}\times\mathscr{U}$ with the following properties:
(a) $b(x,u)$ is upper semi-continuous with respect to x and u (jointly).
(b) For every $x\in\mathscr{X}$ there is a $u\in\mathscr{U}$ such that $b(x,u)\geqq 0$.
(c) $b(x,u)$ is nonincreasing with respect to $u\in\mathscr{U}$.
(d) There is a $g\in R^d$, $g\neq 0$, such that for all $\alpha>0$, $x\in\mathscr{X}$, $u\in\mathscr{U}$, $g\neq 0$, there holds $x+\alpha g\in\mathscr{X}$ and $b(x+\alpha g,u)=b(x,u)+\alpha$.

Then $b(x,u)$ is the benefit function, with reference vector g, of an upper semi-continuous utility function on $\mathscr{X}$ with range $\mathscr{U}$. [That is, $b(x,u)$ is really $b(g;\, x,u)$.] Furthermore, g is weakly good for the corresponding utility function.

Proof. Let

$$u(x)=\max\{v\colon b(x,v)\geqq 0, v\in\mathscr{U}\}.$$

By property (b) the maximization operation is not void. By the upper semi-continuity of $b(x,u)$ with respect to u, and the compactness of $\mathscr{U}$ the maximum exists. We show that the function u defined above is upper semi-continuous. Suppose it were not. Then there is $\varepsilon>0$ and a sequence $\{x_i\}$ with $x_i\in\mathscr{X}$, $x_i\to\bar{x}$ such that $u(x_i)>u(\bar{x})+\varepsilon$ for all i. Let $u_i=u(x_i)$. Since the u_i's are contained in a bounded interval, there is a limit point $\bar{u}$. Without loss of generality, we can assume $u_i\to\bar{u}$. We have $b(x_i,u_i)\geqq 0$ for all i. Since b is upper semi-continuous, it follows that $b(\bar{x},\bar{u})\geqq 0$. Therefore, by definition $u(\bar{x})\geqq\bar{u}$. However, since $u(x_i)>u(\bar{x})+\varepsilon$ for all i, we must have $\bar{u}\geqq u(\bar{x})+\varepsilon$, which is a contradiction.

Next we show that b is the benefit function of u. Let $\hat{b}$ be the benefit function of u (with reference g). First assume $\hat{b}(x,u) > -\infty$. Then

$$\hat{b}(x,u) = \max\{\beta\colon u(x-\beta g) \geqq u\}$$

$$= \max\{\beta\colon \max\{\upsilon\colon b(x-\beta g,\upsilon) \geqq 0\} \geqq u\}$$

$$= \max\{\beta\colon \max\{\upsilon\colon b(x,\upsilon) \geqq \beta\} \geqq u\}.$$

In the maximization with respect to β we note that $\beta = b(x,u)$ is feasible, since then $\upsilon = u$ is feasible for the inner maximization. Thus $\hat{b}(x,u) \geqq b(x,u)$.

Now take $\beta > b(x,u)$ in the inner maximization. Then υ must be selected so that $b(x,\upsilon) > b(x,u)$. By the nonincreasing nature of $b(x,\upsilon)$, it follows that $\upsilon < u$, and this is not feasible for the outer maximization. Therefore, $\hat{b}(x,u) \leqq b(x,u)$. Hence, together with the above we have $\hat{b}(x,u) = b(x,u)$.

Now consider the case $\hat{b}(x,u) = -\infty$. This means that $u(x-\beta g) < u$ for all β. This in turn means by definition of $u(x-\beta g)$, that

$$\max\{\upsilon\colon b(x-\beta g,\upsilon) \geqq 0\} < u$$

for all β. Equivalently, $\upsilon \geqq u$ implies $b(x-\beta g,\upsilon) < 0$, which by the translation property means $\upsilon \geqq u$ implies $b(x,\upsilon) < \beta$ for all β. Hence $b(x,\upsilon) = -\infty$ for all $\upsilon \geqq u$. In particular $b(x,u) = -\infty$.

Now we show that g is weakly good for this utility function. Suppose $x \in \mathscr{X}$, and $\alpha > 0$. Then

$$u(x+\alpha g) = \max\{\upsilon\colon b(x+\alpha g,\upsilon) \geqq 0\}$$

$$= \max\{\upsilon\colon b(x,\upsilon) + \alpha \geqq 0\}.$$

Let $\bar{u} = u(x), \bar{\upsilon} = u(x+\alpha g)$. We have $b(x,\bar{u}) \geqq 0$ and $\bar{\upsilon} = \max\{\upsilon\colon b(x,\upsilon) \geqq -\alpha\}$. Thus $\bar{\upsilon} \geqq \bar{u}$, which shows that g is weakly good. □

2.2. *Aggregate benefits*

Formally, we can define the aggregate benefit function for a group of consumers by summing their individual benefits. For example, suppose there are n consumers. Each consumer i has a consumption possibility set $\mathscr{X}_i$ with the properties discussed in section 2 and a continuous utility function u_i. Then the aggregate benefit function for this group of consumers is by definition equal to the maximum amount of the bundle g that the group would trade to move from existing utility levels $U = (u_1, u_2, \ldots u_n)$ to the allocation $X = (x_1, x_2, \ldots, x_n)$. That is, the benefit function for the group is

$$B(g;\, X, U) = \max \sum_{i=1}^{n} \beta_i \quad \text{sub to } u_1(x_1 - \beta_1 g) \geqq u_1,$$
$$u_2(x_2 - \beta_2 g) \geqq u_2,$$
$$\vdots$$
$$u_n(x_n - \beta_n g) \geqq u_n,$$

where $B(g;\, X, U) = -\infty$ is understood if the above constraints are not feasible.

It is clear that the maximization problem above separates into n problems. That is we see that

$$B(g;\, X, U) = \sum_{i=1}^{n} b_i(g;\, x_i, u_i).$$

Thus, the benefit function of the group is simply the sum of the individual benefit functions. That is, benefit functions can be meaningfully summed across consumers to obtain a measure of aggregate benefits.

3. Examples

In this section, we compute the benefit function for some specific functional forms.

Example 1 (*Cobb–Douglas*). Consider the Cobb–Douglas utility function

$$u(x_1, x_2, \ldots, x_m) = \prod_{i=1}^{m} x_i^{\alpha_i},$$

where $\alpha_i > 0, i = 1, 2, \ldots, m$. For general $g > 0$ the benefit function cannot be expressed in closed-form. However, for

$$g = (0, 0, \ldots, 1, 0, \ldots, 0)$$

with the 1 in the jth position, the benefit function is found by solving

$$(x_j - \beta)^{\alpha_j} \prod_{i \neq j} x_i^{\alpha_i} = u.$$

The resulting β is equal to $b(x, u)$. Hence

$$b(x,u)=x_j-\left[\frac{u}{\prod_{i\neq j}x_i^{\alpha_i}}\right]^{1/\alpha_j}.$$

Example 2 (*Cobb–Douglas in logarithmic form*). For

$$u(x_1,x_2,\dots,x_n)=\sum_{i=1}^{n}\alpha_i\ln x_i,$$

and $g=(0,0,\dots,1,0,\dots,0)$ as in Example 1, we solve

$$\alpha_j\ln(x_j-\beta)+\sum_{i\neq j}\alpha_i\ln x_j=u.$$

Hence

$$b(x,u)=x_j-\exp\left\{\frac{u-\sum_{i\neq j}\alpha_i\ln x_i}{\alpha_j}\right\}.$$

Example 3 (*Leontief utility*). Consider the utility function

$$u(x_1,x_2,\dots,x_m)=\min\left\{\frac{x_1}{a_1},\frac{x_2}{a_2},\dots,\frac{x_m}{a_m}\right\}$$

and the reference good

$$g=(1,1,\dots,1).$$

The benefit function is found by solving $u(x-\beta g)=u$. Or, equivalently,

$$\min\left\{\frac{x_1-\beta}{a_1},\frac{x_2-\beta}{a_2},\dots,\frac{x_n-\beta}{a_m}\right\}=u.$$

This gives

$$b(x,u)=\min\{x_1-a_1u,x_2-a_2u,\dots,x_m-a_mu\}.$$

Example 4 (*additive*). Suppose

$$u(x_1,x_2,\dots,x_n)=f_1(x_1)+f_2(x_2)+\cdots+f_n(x_n),$$

where each f_i is concave and increasing. Let us take $g=(1,0,\ldots,0)$. Then the benefit function is found by solving

$$f_1(x_1-b)+f_2(x_2)+\cdots+f_n(x_n)=u.$$

Hence

$$b(x,u)=x_1-f_1^{-1}\{u-f_2(x_2)-f_3(x_3),\ldots,f_n(x_n)\}.$$

The logarithmic form of the Cobb–Douglas utility (Example 2) is an example of this type.

Another special case of interest is

$$u(x_1,x_2,x_3)=-\frac{1}{x_1}-\frac{1}{x_2}-\frac{1}{x_3}.$$

This gives

$$b(x,u)=x_1+\frac{1}{u-1/x_2-1/x_3}=x_1+\frac{x_2x_3}{ux_2x_3+x_2+x_3}.$$

Example 5 (the oriented–quadratic utility). We propose here a special kind of degenerate quadratic utility that is both mathematically convenient and economically meaningful. We refer to this form as the oriented–quadratic form.

Let the reference bundle $g\geqq 0$, $g\neq 0$, be given. Select d satisfying[1] $d^{\mathrm{T}}g=1$ and select an $n\times n$ symmetric matrix B that is negative semi-definite (of rank $n-1$) with $Bg=0$. Then define

$$u(x)=d^{\mathrm{T}}x+\tfrac{1}{2}x^{\mathrm{T}}Bx.$$

This utility function is *oriented* in the direction g. The translation property of the utility function is easily verified as

$$\begin{aligned}u(x+\alpha g)&=d^{\mathrm{T}}(x+\alpha g)+\tfrac{1}{2}(x+\alpha g)^{\mathrm{T}}B(x+\alpha g)\\&=d^{\mathrm{T}}x+\alpha d^{\mathrm{T}}g+\tfrac{1}{2}x^{\mathrm{T}}Bx+\alpha g^{\mathrm{T}}Bx+\tfrac{1}{2}\alpha^2 g^{\mathrm{T}}Bg\\&=d^{\mathrm{T}}x+\alpha+\tfrac{1}{2}x^{\mathrm{T}}Bx\\&=u(x)+\alpha.\end{aligned}$$

[1]In this example we consider all vectors to be column vectors and let d^{T} denote the transpose. This clarifies the equations that have products of vectors times matrices.

The corresponding benefit function is found from the equation

$$d^{\mathrm{T}}(x-bg)+\tfrac{1}{2}(x-bg)^{\mathrm{T}}B(x-bg)=u.$$

Or (using the translation property found above)

$$d^{\mathrm{T}}x-b+\tfrac{1}{2}x^{\mathrm{T}}Bx=u.$$

Hence

$$b(x,u)=d^{\mathrm{T}}x+\tfrac{1}{2}x^{\mathrm{T}}Bx-u.$$

We see that the utility function and the benefit function have essentially identical expressions in the oriented–quadratic case. The reason is that the indifference curves are equally spaced.

Example 6 (general oriented utility). Suppose

$$u(x)=d^{\mathrm{T}}x+F(x)$$

where $d^{\mathrm{T}}g=1$, $F(x+\alpha g)=F(x)$ for all α, and F is concave. Then, it is readily deduced that

$$b(x,u)=d^{\mathrm{T}}x+F(x)-u.$$

4. Duality properties

We come now to what is one of the most important properties of the benefit function; namely, its dual relation with the expenditure function. We show that the benefit function is the precursor of the expenditure function, in the sense that the expenditure function is (in a certain sense) the dual of the benefit function. This leads naturally to a definition of a second dual – the dual of the expenditure function – which is termed the hyper-benefit function. A central result of this section establishes the full duality of these concepts by showing that under appropriate conditions the hyper-benefit function is, in fact, identical to the benefit function.

Definition. Given a utility function u on $\mathscr{X}$, the *expenditure function* is

$$e(p,u)=\inf_{x\in\mathscr{X}}\{p\cdot x: u(x)\geqq u\},$$

where $p\in R^m$ and $u\in\mathscr{U}$.

Note that if the function u is continuous and if $p>0$, then the infimum in the definition will be achieved, and hence can be replaced by minimization. It is well known that the expenditure function enjoys a number of useful properties.

Proposition 8. The expenditure function satisfies:
(*a*) *Homogeneity*: $e(tp,u)=te(p,u)$ *for all* $t\geqq 0$.
(*b*) *Monotonicity*: $e(p,u)$ *is nondecreasing in* u.
(*c*) *Concavity*: $e(p,u)$ *is a concave function with respect to* p.
(*d*) *Continuity*: *as a function of* p, $e(p,u)$ *is upper semi-continuous on the interior of the region where it is finite.*

Proof. See Diewert (1974) and Berge (1963). □

We now derive an alternate characterization of the expenditure function that shows its relation to the benefit function. [Throughout the following we frequently write $b(x,u)$ for the value of the benefit function, suppressing the dependence on g.]

Proposition 9. Suppose $p\in R^m$ *with* $p\cdot g>0$. *Then*

$$e(p,u)=\inf_{x\in\mathscr{X}}\{p\cdot x-b(x,u)p\cdot g\}.$$

Proof. Fix $u\in\mathscr{U}$ and $p\in R^m$ with $p\cdot g>0$. Given any $x\in\mathscr{X}$, suppose first that $b(x,u)>-\infty$. We have $u(x-b(x,u)g)\geqq u$. Hence,

$$p\cdot(x-b(x,u)g)\geqq e(p,u).$$

Alternatively, if $b(x,u)=-\infty$, then clearly $p\cdot x-b(x,u)p\cdot g\geqq e(p,u)$ by default. Hence, in either case, $p\cdot x-b(x,u)p\cdot g\geqq e(p,u)$. Thus

$$\inf_x\{p\cdot x-b(x,u)pg\}\geqq e(p,u).$$

To show the converse, fix $\varepsilon>0$ and suppose $x\in\mathscr{X}$ is such that $u(x)\geqq u$ and $p\cdot x\leqq e(p,u)+\varepsilon$; (that is, x is close to achieving the infimum in the definition of $e(p,u)$.) By Proposition 2(c), $b(x,u)\geqq 0$. Also since $p\cdot g>0$, we have

$$p\cdot x-b(x,u)p\cdot g\leqq p\cdot x\leqq e(p,u)+\varepsilon.$$

Since $\varepsilon>0$ was arbitrary,

$$\inf_{x \in \mathscr{X}} \{p \cdot x - b(x, u) p \cdot g\} \leqq e(p, u). \qquad \square$$

This alternate characterization of $e(p, u)$ applies only if $p \cdot g > 0$. This would include all $p \geqq 0$ if $g > 0$. In general, prices with $p \cdot g \leqq 0$ are not needed, and hence the apparent restriction is of no real consequence.

The factor $p \cdot g$ in the second term of the above representation can be eliminated by simply setting it to 1. This essentially restricts the p vectors under consideration to those satisfying $p \cdot g = 1$. This, of course, is the standard form of normalization used in many economic arguments. We then obtain (trivially) the following corollary, which displays the relation between b and e in a neater form.

Corollary 1. If $p \cdot g = 1$, then

$$e(p, u) = \inf_{x \in \mathscr{X}} \{p \cdot x - b(x, u)\}. \tag{1}$$

The relation between the benefit function b and the expenditure function e can be illustrated graphically in a simple and intuitive way. For the construction in two dimensions, it is simplest to take $g = (1, 0)$, corresponding to the direction of the horizontal axis. With this choice, if a price vector p is normalized with $p \cdot g = 1$ (that is, $(p)_1 = 1$), then the constant c associated with the hyperplane $H = \{x : p \cdot x = c\}$ is just the distance from the origin to the point where the hyperplane intersects the horizontal axis. Thus, given x and p, the value $p \cdot x$ is measured by the point where the hyperplane through x intersects the horizontal axis.

The alternative characterizations of the expenditure function are shown in fig. 1. Usually the value of the expenditure function is found by finding the point x^* in the upper contour set $S = \{x : u(x) \geqq u\}$ which has the least value of $p \cdot x$. This point will be that where a p-hyperplane is tangent to S. The value of the expenditure function is then $p \cdot x^*$. If $p \cdot g = 1$, this value will be equal to the distance from the origin to the point where the p-hyperplane intersects the horizontal axis.

The formula given by the corollary above does not restrict the search to x's inside of S. A point such as x^1 in the figure will also yield a value of $p \cdot x - b(g; x, u)$ equal to $e(p, u)$. While a point such as x^2 will lead to a larger value. The new formulation introduces a bit of degeneracy, in the sense that many x's (namely all those on the horizontal line through x^*) will achieve the infimum in (1).

To make the duality between b and e more explicit, we introduce the dual of e.

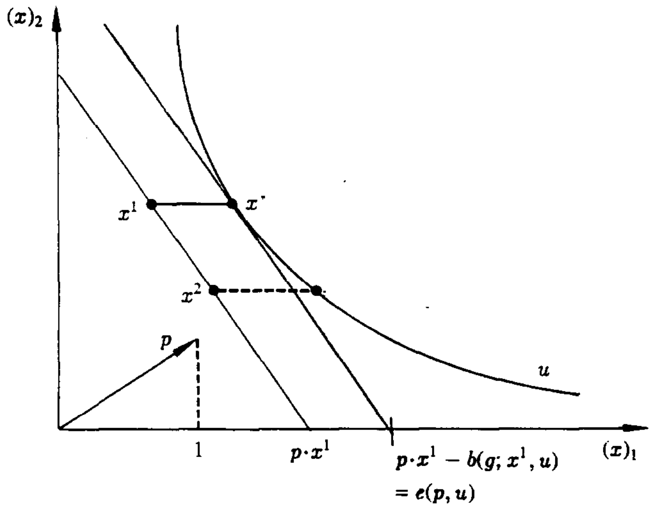

Fig. 1. Alternative definition of expenditure function.

Definition. Corresponding to the expenditure function e, we define the *hyper-benefit function*

$$\bar{b}(g;\ x, u) = \inf\{p \cdot x - e(p, u) : p \cdot g = 1\} \tag{2}$$

for any $g \in R^m_+, g \neq 0, x \in \mathscr{X}$ and $u \in \mathscr{U}$.

The formula is, of course, the dual of (1). It is not surprising, therefore, that we have the following.

Proposition 10. The hyper-benefit function satisfies:
(*a*) *Translation: if $x \in \mathscr{X}$ and $x + \alpha g \in \mathscr{X}$, then $\bar{b}(g;\ x + \alpha g, u) = \alpha + \bar{b}(g;\ x, u)$.*
(*b*) *Monotonicity: $\bar{b}(g;\ x, u)$ is nondecreasing in u.*
(*c*) *Concavity: $\bar{b}(g;\ x, u)$ is concave with respect to x.*
(*d*) *Continuity: $\bar{b}(g;\ x, u)$ is continuous with respect to x in the interior of the region in which it is finite.*

Proof. (a)

$$p \cdot (x + \alpha g) - e(p, u) = p \cdot x + \alpha p \cdot g - e(p, u).$$

Hence, if $p \cdot g = 1$,

$$p\cdot(x+\alpha g)-e(p,u)=\alpha+p\cdot x-e(p,u).$$

Taking the infimum over p gives the result.
(b) This follows because for any p, the function $p\cdot x-e(p,u)$ is nonincreasing in u (by Proposition 8b).
(c) Fix $u\in\mathscr{U}$. Given $x_1,x_2\in\mathscr{X}$ and α, $0\leqq\alpha\leqq 1$,

$$\begin{aligned}
&\bar{b}(g;\ \alpha x_1+(1-\alpha)x_2,u)\\
&\quad=\inf\{p\cdot(\alpha x_1+(1-\alpha)x_2)-e(p,u): p\cdot g=1\}\\
&\quad\geqq\inf\{\alpha p\cdot x_1-\alpha e(p,u): p\cdot g=1\}\\
&\qquad+\inf\{(1-\alpha)p\cdot x_1-(1-\alpha)e(p,u): p\cdot g=1\}\\
&\quad=\alpha\bar{b}(g;x_1,u)+(1-\alpha)\bar{b}(g;\ x_2,u).
\end{aligned}$$

(d) This follows from part (c). □

We now come to the main duality result, showing that the hyper-benefit function is the dual of the dual of the benefit function itself. The hyper-benefit function, being defined by hyperplanes, essentially uses only the convexified version of the original utility function. If the original utility function is quasi-concave, then the hyper-benefit function will agree with the benefit function. (Note 1: The proof of the following main theorem could be simplified if either g was good, the utility function was strongly monotonic, or $g>0$. It is, perhaps, surprising that in fact none of these conditions is required.) (Note 2: This theorem is similar to the standard duality theorem for conjugate duality. However, because our definition of duality is a degenerate version of conjugate duality, a separate proof is required.)

Theorem 1. Assume the utility function u is quasi-concave and continuous. Then $\bar{b}(g;\ x,u)=b(g;x,u)$ for all $g\in R^m_+$, $g\neq 0$, $x\in\mathscr{X}$ and $u\in\mathscr{U}$.

Proof. If $b(g;\ x,u)>-\infty$, we have $u(x-b(g;\ x,u)g)\geqq u$. Hence by definition of e, it follows that $e(p,u)\leqq p\cdot g-b(g;\ x,u)p\cdot g$. Rewriting this we have $p\cdot x-e(p,u)\geqq b(g;\ x,u)p\cdot g$. Therefore

$$\bar{b}(g;\ x,u)\geqq b(g;\ x,u).$$

This is clearly also true if $b(g;\ x,u)=-\infty$.

To prove the reverse inequality fix $x\in\mathscr{X}$. Assume first that $b(g;\ x,u)>-\infty$. Let $x_0=x-b(g;\ x,u)g$. Then x_0 is a boundary point of the upper contour set $S=\{x':u(x')\geqq u, x'\in\mathscr{X}\}$. S is a closed set by continuity.

Given $\varepsilon>0$, consider the point $z\in R^m$ defined by $z=x_0-\varepsilon g$. (This point

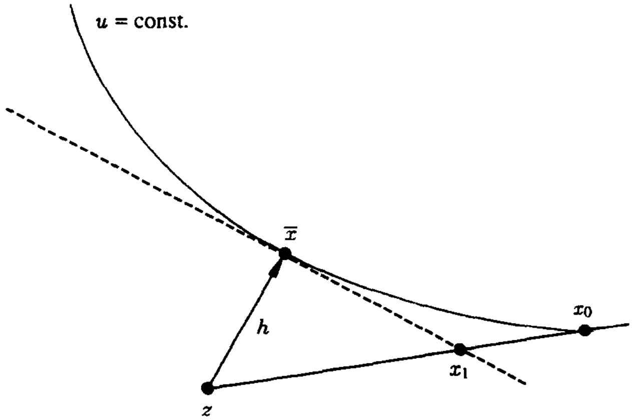

Fig. 2. Construction for proof.

may not be feasible, but that does not matter.) Clearly $z \notin S$. Let $\bar{x}$ be the point in S that is closest to z as measured by the Euclidean norm, and let h be the vector $\bar{x}-z$. (See fig. 2.) The hyperplane defined by $\{x': h\cdot x' = h\cdot \bar{x}\}$ is a supporting hyperplane of S at $\bar{x}$, since it is orthogonal to h. We have $h\cdot z < h\cdot \bar{x}$, $h\cdot x_0 \geqq h\cdot \bar{x}$. Hence, $h\cdot(x_0 - z) > 0$. But using $x_0 - z = \varepsilon g$, this gives $h\cdot g > 0$. We now let $p = h/(h\cdot g)$, so that $p\cdot g = 1$. Thus p is a feasible vector for the infimum operation defining $b(g;\, x, u)$.

Now for the p above, $e(p,u) = p\cdot\bar{x}$. Hence,

$$\begin{aligned}\bar{b}(g;\, x, u) &\leqq p\cdot x - p\cdot \bar{x}\\ &= p\cdot x - p\cdot x_0 + p\cdot x_0 - p\cdot \bar{x}\\ &= p\cdot g\, b(g;\, x, u) + p\cdot x_0 - p\cdot \bar{x}\\ &= b(g;\, x, u) + p\cdot(x_0 - \bar{x}),\end{aligned}$$

where in the second line we used $x_0 = x - b(g;\, x, u)g$. We need to bound the term $p\cdot(x_0 - \bar{x})$.

Since $h\cdot x_0 \geqq h\cdot \bar{x}$ and $h\cdot z < h\cdot \bar{x}$, there is a (unique) point x_1 on the segment joining these two which is in the supporting hyperplane. (See fig. 2.) We then write

$$p\cdot(x_0 - \bar{x}) = p\cdot(x_0 - x_1) + p\cdot(x_1 - \bar{x}).$$

The second term on the right is zero since both x_1 and $\bar{x}$ are in the

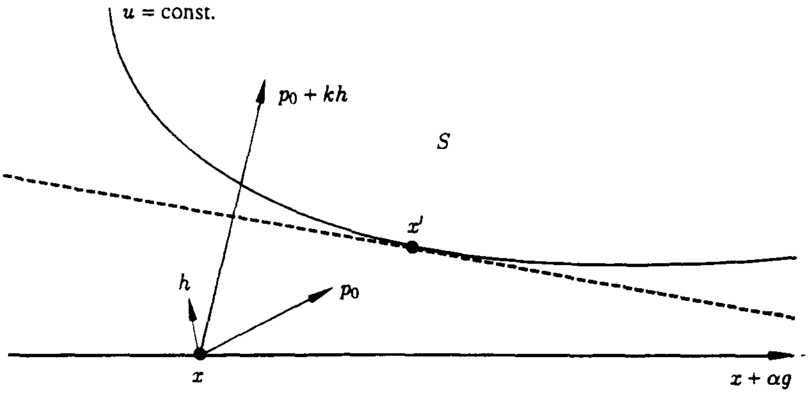

Fig. 3. Case where $x+\alpha g$ bounded away from S.

hyperplane. The first term is less than or equal to ε since $x_0 - x_1$ is a shortened version of $x_0 - z = \varepsilon g$. Using this above, we have

$$\bar{b}(g;\ x, u) \leqq b(g;\ x, u) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we have $\bar{b}(g;\ x, u) \leqq b(g;\ x, u)$. Combining this with the reverse inequality found earlier gives equality.

We must now treat the case $b(g;\ x, u) = -\infty$. This means that the ray $Q = \{x' : x' = x + \alpha g,\ \alpha \text{ real}\}$ does not intersect the upper contour set $S = \{x' : u(x') \geqq u\}$. Suppose first that this ray is bounded away from S. Then the set $S - Q$ is convex and bounded away from zero. Let h be the vector in the closure of $S - Q$ closest to 0. Then there is $\delta > 0$ such that $h \cdot s - h \cdot q \geqq \delta$ for all $s \in S$, $q \in Q$. In particular for any $\bar{x} \in S$ there holds $h \cdot \bar{x} - h \cdot (x + \alpha g) > \delta$ for all α. This implies $h \cdot g = 0$. In other words, h defines a hyperplane containing the ray, which can be shifted outward by an amount δ before it touches S. Now let $p_0 > 0$ be a vector with $p_0 \cdot g = 1$. Then define $p = p_0 + kh$. (See fig. 3.) Clearly $p \cdot g = 1$. We have

$$\begin{aligned} e(p, u) &= \inf\{p \cdot x' : x' \in S\} \\ &= \inf\{p_0 \cdot x' + kh \cdot x' : x' \in S\} \\ &\geqq e(p_0, u) + kh \cdot x + k\delta. \end{aligned}$$

Therefore

$$\bar{b}(g;\ x, u) \leqq p \cdot x - e(p, u)$$

$$\leqq (p_0 + kh) \cdot x - e(p_0, u) - kh \cdot x - k\delta$$

$$\leqq p_0 \cdot x - e(p_0, u) - k\delta.$$

We have $e(p_0, u) > -\infty$ since $p_0 > 0$ and $\mathscr{X}$ is bounded below. Since k was arbitrary, $\bar{b}(g;\, x, u) = -\infty$.

Now consider the general case where $b(g;\, x, u) = -\infty$ and the ray $x + \alpha g$ is not bounded away from S. Let $z = x + kg$, for some $k > 0$, and let $\bar{x}$ be the point in S closest to z. Finally let $h = \bar{x} - z$. The hyperplane $H = \{x' : h \cdot x' = h \cdot \bar{x}\}$ is a supporting hyperplane for S at $\bar{x}$. We have $h \cdot z = h \cdot \bar{x} - \delta$ for some $\delta > 0$. On the other hand, $z + \alpha g$ is arbitrarily close to S for sufficiently large α. Thus given $\varepsilon > 0$, we have $h \cdot (z + \alpha g) \geqq h \cdot \bar{x} - \varepsilon$ for large α. It follows that $h \cdot g > 0$.

Let $p = h / h \cdot g$. Then $p \cdot g = 1$. Also $e(p, u) = p \cdot \bar{x}$. Hence

$$\bar{b}(g;\, x, u) \leqq p \cdot x - p \cdot \bar{x}$$

$$\leqq p \cdot x - p \cdot (x + kg + h)$$

$$\leqq -kp \cdot g - p \cdot h < -k.$$

Since k was arbitrary $\bar{b}(g;\, x, u) = -\infty$ and the proof is complete. □

4.1. Relation to the distance function

Another function, the *distance function*, [see Shephard (1953)] is of common use in microeconomic theory and it also has a dual relation to the expenditure function. It is worthwhile to briefly compare it with the benefit function. The distance function d is defined as

$$d(x, u) = \max\{\delta : u(x/\delta) \geqq u\}.$$

Geometrically $d(x, u)$ is the maximum factor by which x can be reduced in order to reach the indifference curve determined by u.

For $p \geqq 0$, the expenditure function can be written in terms of the distance function as

$$e(p, u) = \min_{x \in \mathscr{X}} \{p \cdot x : d(x, u) = 1\}.$$

And, under appropriate assumptions, the distance function can be recovered as the dual of the dual by

$$d(x,u)=\min_{p\geqq 0}\{p\cdot x: e(p,u)=1\}.$$

These are strong dual relations, and indicate that the distance function is also a natural precursor of the expenditure function.

There are, however, important distinctions between distance–expenditure duality and the benefit–expenditure duality. Mathematically, the distinction is simply that the distance function is based on Lagrange duality, while the benefit function is based on a modified Legendre or Fenchel duality. [See Fenchel (1953).] But, from an economic viewpoint the main distinction between the two is traced to a difference in the method of price vector normalization. The distance function is defined by normalizing the expenditure of the consumer to 1. The benefit function is defined by normalizing prices absolutely, by $p\cdot g=1$. The advantage of the second becomes apparent when one considers problems involving more than one consumer. The single normalization of the benefit function theory can be applied to all consumers, while the distance function approach requires that a given price vector be normalized differently for each consumer. The original (primal) definition of the distance function as a scale factor also has no ready welfare interpretation. Hence, although there are indeed connections between the benefit function and the distance function, these connections are somewhat convoluted. The distance function can be useful in developing relations in individual consumer theory. The benefit function has use in developing group welfare relations.

5. Conclusions

The benefit function is a natural generalization of the familiar concept of willingness to pay, but measured with respect to an arbitrary bundle of goods. This generalization has the advantage that, unlike a measure that uses a specific single reference commodity, the benefit function can be described in terms of the natural variables of a situation, rather than in terms of partitioned variables. This makes the benefit function a natural candidate for deeper study. This paper shows that the benefit function has desirable structural and continuity properties. Its strong economic interpretation and its duality relation with the expenditure function make it a valuable general tool for economic analysis – especially for welfare issues.

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