The Exponential Formula

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Note: We may need to relabel the vertices so as to produce a *standard labeling*. By this we mean that for a graph of n vertices, the label set is $[n]$.

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  $\Rightarrow$ There are 8 vertex labeled undirected graphs of three vertices.

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- This is precisely the question that the exponential formula will answer for us.

Terminology

Definition: A *card* $C(S, p)$ is a pair consisting of a fintite set S (the label set) of positive integers, and a picture $p \in P$. The *weight* of C is $n = |S|$. A card of weight n is called *standard* if its label set is $[n]$.

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Definition: A *hand* H is a set of cards whose label sets form a partition of $[n]$, for some n. In other words, if n denotes the sum of the weights of the cards in the hand, then the label sets of the cards are pairwise disjoint, nonempty, and their union is $[n]$.

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Definition: The *weight of a hand* is the sum of the weights of the cards in the hand.

Definition: A *relabeling* of a card $C(S, p)$, with a set S' is defined if $|S| = |S'|$, and it is the card $C(S', p)$. If $S' = [|S|]$ then we have the *standard* relabeling of the card.

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Definition: A *deck* $\mathcal{D}$ is a finite set of standard cards whose weights are all the same and whose pictures are all different. The *weight of the deck* is the common weight of all of the cards in the deck.

Definition: An *exponential family* $\mathcal{F}$ is a collection of decks $\mathcal{D}_1, \mathcal{D}_2, ...$ where for each $n = 1, 2, ...$ the deck $\mathcal{D}_n$ is of weight n.

Notation

- We denote the number of cards in deck $\mathcal{D}_n$ by d_n and call $\mathcal{D}(x)$ the exponential generating function of $\{d_n\}_1^\infty$, the deck enumerator of the family.

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- If $h(n) = \sum_k h(n,k)$ is the number of hands of weight n, we write $\mathcal{H}(x)$ for the EGF of $\{h(n)\}$, instead of $\mathcal{H}(x,1)$. Thus, our question asks for some relationship between $\mathcal{H}(x,y)$ and $\mathcal{D}(x)$.

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Hands: Recall that a hand is a collection of cards whose label sets partition $[n]$, where n is the weight of the hand. For $\mathcal{F}_1$ this means a hand $\mathcal{H}$ corresponds to a not necessarily connected graph with standard labels.

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- Hence, our main question of interest asks for the relationship between the numbers of all labeled graphs and all connected labeled graphs of all sizes.

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Note: $h(n, k)$ here is an object we have studied before. Namely, $h(n, k) = s(n, k)$, a stirling number of the first kind. The exponential formula can aid us in studying these objects.

The Main Counting Theorems

Merging Two Exponential Familes

Let $\mathcal{F}'$ and $\mathcal{F}''$ be two exponential families whose picture sets, $\mathcal{P}'$ and $\mathcal{P}''$ are disjoint. We form a third family, $\mathcal{F}$, and write $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ as follows:

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Fix $n \geq 1$. From $\mathcal{F}'$ we take all of the d_n' cards of deck $\mathcal{D}_n'$ and put them in a new pile. Then from $\mathcal{F}''$ we take all d_n'' of its cards from deck $\mathcal{D}_n''$ and add them to the pile which now contains $d_n' + d_n'' = d_n$ different cards.

The Fundamental Lemma of Labeled Counting

Let $\mathcal{F}', \mathcal{F}''$ be two exponential familes and $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ be their merger. Further, let $\mathcal{H}'(x, y), \mathcal{H}''(x, y)$, and $\mathcal{H}(x, y)$ be their respective two-variable hand enumerators of these families. Then

$$\mathcal{H}(x, y) = \mathcal{H}'(x, y)\mathcal{H}''(x, y)$$

Proof

Consider hand H in $\mathcal{F}$. Some of the cards in H came from $\mathcal{F}'$, and others came from $\mathcal{F}''$. The collection that came from $\mathcal{F}'$ forms a sub-hand, call it H' of weight n' and having k' cards that have been relabled in an order preserving way with a certain label set $S \subset [n]$. All hands H in the merged family $\mathcal{F}$ are uniquely determined by H', the choice of new labels S, and the renaming sub-hand H'' with the labels of $[n] - S$. Therefore, the number of hands in the merged family $\mathcal{F}$ that have weight n and have exactly k cards is

$$\begin{aligned} h(n,k) &= \sum_{n',k'} \binom{n}{n'} h'(n',k') h''(n-n', k-k') \\ &= \left[\frac{x^n}{n!} y^k\right] \mathcal{H}'(x,y)\mathcal{H}''(x,y) \end{aligned}$$

Theorem

Let $\mathcal{F}$ be an exponential family whose deck and hand enumerators are $\mathcal{D}(x)$ and $\mathcal{H}(x,y)$, respectively. Then

$$\mathcal{H}(x,y) = e^{y\mathcal{D}(x)}$$

In detail, the number of hands of weight n and k cards is

$$h(n,k) = \left[\frac{x^n}{n!}\right]\left(\frac{\mathcal{D}(x)^k}{k!}\right)$$

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$$\sum_{\substack{p_1,\ldots,p_n \geq 0 \\ p_1+2p_2+\cdots+np_n=n \\ p_1+\cdots+p_n=k}} \frac{n!}{p_1!p_2!\cdots p_n!(1!)^{p_1}(2!)^{p_2}\cdots(n!)^{p_n}} d_1^{p_1} d_2^{p_2} \cdots d_n^{p_n}$$

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$$= n! \sum_{\substack{p_1,\ldots,p_n\geq 0\\ p_1+2p_2+\cdots+np_n=n\\ p_1+\cdots+p_n=k}} \frac{1}{p_1!p_2!\cdots p_n!}\left(\frac{d_1}{1!}\right)^{p_1}\left(\frac{d_2}{2!}\right)^{p_2}\cdots\left(\frac{d_n}{n!}\right)^{p_n}$$

The Exponential Formula

$$= \frac{n!}{k!} \sum_{\substack{p_1,\ldots,p_n \geq 0 \\ p_1+\cdots+p_n=k}} \frac{k!}{p_1!p_2!\cdots p_n!} \left(\frac{d_1 x}{1!}\right)^{p_1} \left(\frac{d_2 x^2}{2!}\right)^{p_2} \cdots \left(\frac{d_n x^n}{n!}\right)^{p_n} \Bigg|_{x^n}$$

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$$= \frac{n!}{k!} \left(\frac{d_1 x}{1!} + \frac{d_2 x^2}{2!} + \cdots + \frac{d_n x^n}{n!} \cdots \right)^k \Bigg|_{x^n} \quad \text{multinomial theorem}$$

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$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} h(n,k) y^k$$

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$$= \sum_{n=0}^{\infty}x^n\left(\sum_{k=0}^{n}\frac{y^k}{k!}\left(\sum_{m=0}^{\infty}\frac{d_m x^m}{m!}\right)^k\Bigg|_{x^n}\right)$$

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&= \sum_{n=0}^{\infty} x^n \left(\sum_{k=0}^{n} \frac{y^k}{k!} \left(\sum_{m=0}^{\infty} \frac{d_m x^m}{m!} \right)^k \Bigg|_{x^n} \right) \\
&= \sum_{k=0}^{\infty} \frac{y^k}{k!} \mathcal{D}(x)^k
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&= \sum_{k=0}^{\infty} \frac{y^k}{k!} \mathcal{D}(x)^k \\
&= e^{y\mathcal{D}(x)} = \mathcal{H}(x,y)
\end{aligned}$$

Corollary 1

Let $\mathcal{F}$ be an exponential family, let $\mathcal{D}(x)$ be the *egf* of the sequence $\{d_n\}_1^\infty$ of sizes of the decks, and let $\mathcal{H}(x)$ be the *egf* of the sequence $\{h_n\}_0^\infty$, where h_n is the number of hands of weight n. Then

$$\mathcal{H}(x) = e^{\mathcal{D}(x)}$$

Corollary 2

Let T be a set of positive integers, let $e_T(x) = \sum_{n \in T} x^n/n!$, and let $h_n(T)$ be the number of hands whose weight is n and whose number of cards belongs to the allowable set T. Then

$$e_T(\mathcal{D}(x)) \text{ is the exponential generating function for } \{h_n(T)\}_0^\infty$$

Example 1: Permutations and Cycles

Question: How many permutations of n letters and k cycles are there?

Cards: A card of weight n will have $[n]$ arranged in a cycle in some order, with a set $\mathcal{S}$ of n positive integers on the card.

Picture of a Card: The picture is simply a relabeling of a cycle of the elements in $\mathcal{S}$ using $[n]$

Decks: $\mathcal{D}_n$ consists of exactly one of each card of weight n.

- Thus, $d_n = (n-1)!$ cards in each deck $\mathcal{D}_n$

Hands: A collection of cards such that their label sets are pairwise disjoint and their union is $[n]$

- i.e. a hand is a representation of a permutation of $[n]$ with disjoint cycles given by the cards.

Finding the EGF and Applying the Exponential Formula

We know $d_n = (n-1)!$ so,

$$\mathcal{D}(x) = \sum_{n=1}^{\infty} (n-1)! \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

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Which is the Maclaurin Series for

$$\ln\left(\frac{1}{1-x}\right)$$

Finding the EGF and Applying the Exponential Formula

We know $d_n = (n-1)!$ so,

$$\mathcal{D}(x) = \sum_{n=1}^{\infty} (n-1)! \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Which is the Maclaurin Series for

$$\ln\left(\frac{1}{1-x}\right)$$

Since we found $\mathcal{D}(x) = \ln\left(\frac{1}{1-x}\right)$ we can now apply the exponential formula,

$$\mathcal{H}(x,y) = e^{y\mathcal{D}(x)} = \frac{1}{(1-x)^y}$$

Stirling Numbers of the First Kind

Since $\mathcal{H}(x, y)$ is the generating function for each $h(n, k)$, summing over all k yields the Stirling numbers of the first kind

$$\sum_{k=0}^{n} s(n, k)y^k = \left[\frac{x^n}{n!}\right](1 - x)^{-y}$$

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Calculating the coeffictient of $\frac{x^n}{n!}$ yields

$$n!\binom{y+n-1}{n} = y(y+1)(y+2)\cdots(y+n-1)$$

A Full Generating Function

By the Exponential Formula we know the enumerator of hands of k cards is,

$$\left(\frac{\mathcal{D}(x)^k}{k!}\right) = \frac{1}{k!}\left(\ln\left(\frac{1}{1-x}\right)\right)^k$$

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Hence the Generating function for Stirling numbers of the first kind is

$$s(n,k) = \left[\frac{x^n}{n!}\right]\left(\frac{\left(\ln\frac{1}{1-x}\right)^k}{k!}\right)$$

Set Partitions

Question: How many partitions of n elements into k subsets are there?

Card: For each $n \geq 1$ there is only one card of weight n with the label set $[n]$.

- Thus each deck has exaclty 1 card

Hand: There is hand corresponding to every partition of $[n]$

Generating Function for Stirling Numbers of the Second Kind

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and more specifically

$$h(n, k) = S(n, k) = \left[\frac{x^n}{n!}\right] \left(\frac{(e^x - 1)^k}{k!}\right)$$

Bell Numbers

Recall the Bell numbers were all of the partitions of n with k many subsets, so by summing over all k (i.e. row sums)

$$e^{e^x - 1} = \sum_{k=0}^{\infty} \frac{(e^x - 1)^k}{k!}$$

Subclasses of Permutations

Question: How many permutations of n letters have an even number of cycles and all of the cycles are of odd length?

- We will use the same exponential family as in the example with Stirling numbers of the first kind, however we will only use decks of odd weight.

Generating Functions

Similarly we start by finding the deck enumerator,

$$\mathcal{D}(x) = \sum_{n \text{ odd}} (n-1)! \frac{x^n}{n!}$$
$$= \sum_{r=0}^{\infty} \frac{x^{2r+1}}{2r+1}$$

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Which is the Maclaurin series for

$$\frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) = \ln\left(\sqrt{\frac{1+x}{1-x}}\right)$$

Generating Functions

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By corollary,

$$e_T(x) = \sum_{n \in T} \frac{x^n}{n!}$$

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\begin{aligned}
e_T(x) &= \sum_{n \in T} \frac{x^n}{n!} \\
e_T(x) &= \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!} \\
&= \cosh(x)
\end{aligned}
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By Newton's Binomial theorem we can write $\binom{n}{k}$ any $n \in \mathbb{R}$

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Using Newton's Binomial Theorem

For $\frac{1}{\sqrt{1-x^2}}$ we have $n = -\frac{1}{2}$ so by using the Binomial Theorem,

$$\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} (-1)^k \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{\frac{1}{2}-k+1}{2}\right)\frac{x^{2k}}{k!}$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k}\left(\frac{x}{2}\right)^{2k}$$

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Thus we obtain

$$\binom{n}{\frac{n}{2}}\frac{n!}{2^n}$$

as the number of permutations of n letters with an even number of cycles where each cycle is of odd length.

A Side Note

We can now find the number of permuations of n letters with an *odd* number of cycles where each cycle is of odd length very easily.

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- Simply, instead of using $\cosh(x)$ we use $\sinh(x)$ and continue the same process

2-regular Graphs

Question: How many undirected graphs are there on n vertices where every graph is of degree 2 (i.e. every vertex is connected to 2 other vertices)?

- The graph will be a union of undirected disjoint cycles

We will use the exponental family $\mathcal{F}_1$

2-regular Graphs

- Such a graph only exists if there are at least 3 vertices
- We know that there are $(n-1)!$ directed cycles, thus there are $\frac{(n-1)!}{2}$ undirected cycles, and hence $d_n = \frac{(n-1)!}{2}$.

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Again start by finding the deck enumerator

$$\begin{aligned}\mathcal{D}(x) &= \sum_{n=3}^{\infty} \frac{(n-1)!}{2n!} x^n \\ &= \frac{1}{2}\left(\ln \frac{1}{1-x} - x - \frac{x^2}{2}\right)\end{aligned}$$

By the Exponential Formula,

$$\mathcal{H}(x,y) = \exp\left[\frac{1}{2}\left(\ln\frac{1}{1-x} - x - \frac{x^2}{2}\right)\right]$$
$$= \frac{e^{-\frac{x}{2}-\frac{x^2}{4}}}{\sqrt{1-x}}$$