POMA Solution (Chapter 6)

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Chapter 6
The Riemann-Stieltjes Integral

1. Suppose \( \alpha \) increases on \([a,b]\), \( a \leq x_0 \leq b \), \( \alpha \) is continuous at \( x_0 \), \( f(x_0) = 1 \), and \( f(x) = 0 \) if \( x \neq x_0 \). Prove that \( f \in \mathcal{R}(\alpha) \) and that \( \int f \, d\alpha = 0 \).

**Proof.** \( f \in \mathcal{R}(\alpha) \) follows directly from Theorem 6.10. Clearly we have \( \int f \, d\alpha \geq 0 \) since \( f \) is nonnegative on \([a,b]\). Given \( \varepsilon > 0 \), if \( x_0 = a \) (the case \( x_0 = b \) can be proved similarly), let \( \delta = \min(\varepsilon, \frac{b-a}{2}) > 0 \), and \( P = \{a, a + \delta, b\} \), then

\[
\int f \, d\alpha \leq U(P, f, \alpha) = 1 \times \delta + 0 \times (b - \delta) = \delta \leq \varepsilon.
\]

If \( x_0 \) is an interior point of \([a,b]\), then let \( \delta = \min(\frac{\varepsilon}{2}, \frac{x_0 - a}{2}, \frac{b - x_0}{2}) > 0 \), and \( P = \{a, x_0 - \delta, x_0 + \delta, b\} \), then

\[
\int f \, d\alpha \leq U(P, f, \alpha) = 0 \times (x_0 - \delta) + 1 \times 2\delta + 0 \times (b - x_0 - \delta) = 2\delta \leq \varepsilon.
\]

So in either case we conclude that

\[
0 \leq \int f \, d\alpha \leq \int f \, d\alpha \leq \varepsilon.
\]

Since \( \varepsilon \) is arbitrarily small, it follows that \( \int f \, d\alpha = \int f \, d\alpha = 0 \) (which also gives a direct proof of that \( f \in \mathcal{R}(\alpha) \) without referring to Theorem 6.10), hence \( \int f \, d\alpha = 0 \).
2. Suppose \( f \geq 0, f \) is continuous on \([a, b]\), and \( \int_{a}^{b} f(x) \, dx = 0 \). Prove that \( f(x) = 0 \) for all \( x \in [a, b] \).

Proof. If the conclusion were not true, then there would exist some \( x_0 \in [a, b] \) such that \( f(x_0) > 0 \), without losing of generality, assume \( x_0 \in (a, b) \). Since \( f \) is continuous at \( x_0 \), there exists \( \delta > 0 \) such that \([x_0 - \delta, x_0 + \delta] \subset (a, b)\) and \( f(x) > \frac{f(x_0)}{2} \) for all \( x \in [x_0 - \delta, x_0 + \delta] \), it then follows that

\[
0 = \int_{a}^{b} f(x) \, dx \geq \int_{x_0 - \delta}^{x_0 + \delta} f(x) \, dx > \frac{f(x_0)}{2} \times 2\delta = f(x_0)\delta > 0,
\]

contradiction. Therefore \( f(x) = 0 \) for all \( x \in [a, b] \).

3. Define three functions \( \beta_1, \beta_2, \beta_3 \) as follows: \( \beta_j(x) = 0 \) if \( x < 0 \), \( \beta_j(x) = 1 \) if \( x > 0 \) for \( j = 1, 2, 3 \); and \( \beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = \frac{1}{2} \). Let \( f \) be a bounded function on \([-1, 1]\).

(a) Prove that \( f \in \mathcal{R}(\beta_1) \) if and only \( f(0^+) = f(0) \) and that

\[
\int f \, d\beta_1 = f(0).
\]

(b) State and prove a similar result for \( \beta_2 \).

(c) Prove that \( f \in \mathcal{R}(\beta_3) \) if and only if \( f \) is continuous at 0.

(d) If \( f \) is continuous at 0, prove that

\[
\int f \, d\beta_1 = \int f \, d\beta_2 = \int f \, d\beta_3 = f(0)
\]

Proof.

(a) If \( f(0^+) = f(0) \), given \( \varepsilon > 0 \), there exists \( \delta \in (0, 1/2) \) such that \(|f(x) - f(0)| < \varepsilon/2\) for all \( x \in [0, \delta] \). Take the partition \( P \) of \([-1, 1]\) to be \([-1, 0, \delta, 1]\), and set
\[ M = \sup_{0 \leq x \leq \delta} f(x), \quad m = \inf_{0 \leq x \leq \delta} f(x) \], then

\[ U(P, f, \beta_1) - L(P, f, \beta_1) = (M - m)(\beta_1(\delta) - \beta_1(0)) = M - m \leq 2 \sup_{0 \leq x \leq \delta} |f(x) - f(0)| < \varepsilon. \]

By Theorem 6.6, \( f \in \mathcal{R}(\beta_1) \) on \([-1, 1]\).

Conversely, if \( f \in \mathcal{R}(\beta_1) \) on \([-1, 1]\), given \( \varepsilon > 0 \), by Theorem 6.6, there exists a partition \( P \) of \([-1, 1]\) such that \( U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon \). Suppose such \( P \) consists of \(-1 = x_0, x_1, \ldots, x_n = 1\), and let \( P^* \) be a refinement of \( P \) such that \( 0 \in P^* \). Clearly, there exists \( i \in \{0, \ldots, n-1\} \) such that \( x_i \leq 0 \leq x_{i+1} \). As above, it can be shown \( U(P^*, f, \beta_1) - L(P^*, f, \beta_1) = M - m \), where \( M = \sup_{0 \leq x \leq x_{i+1}} f(x), \quad m = \inf_{0 \leq x \leq x_{i+1}} f(x) \). By Theorem 6.4, we have \( U(P^*, f, \beta_1) - L(P^*, f, \beta_1) \leq U(P, f, \beta_1) - L(P, f, \beta_1) \), thus for all \( x \in [0, x_{i+1}] \):

\[ |f(x) - f(0)| \leq M - m = U(P^*, f, \beta_1) - L(P^*, f, \beta_1) \leq U(P, f, \beta_1) - L(P, f, \beta_1) < \varepsilon. \]

i.e., \( f(0+) = f(0) \).

Under the condition \( f(0) = f(0+) \), with the same argument as above, it can be shown that for any \( \varepsilon > 0 \),

\[ \int f \, d\beta_1 \leq f(0) + \varepsilon, \quad \int f \, d\beta_1 \geq f(0) - \varepsilon. \]

Hence \( f(0) - \varepsilon \leq \int f \, d\beta_1 \leq f(0) + \varepsilon \), since \( \varepsilon \) is arbitrarily small, it follows that \( \int f \, d\beta_1 = \int f \, d\beta_1 = \int f \, d\beta_1 = f(0) \).

(b) The similar result for \( \beta_2 \) is as follows:

\( f \in \mathcal{R}(\beta_2) \) if and only if \( f(0-) = f(0) \) and that \( \int f \, d\beta_2 = f(0) \).

The proof of this statement is almost identical to that of part (a), so we omit it here.

(c) The proof is again very similar to that of part (a): it is sufficient to add another point into the partition and proceed.
(d) Trivial in view of the results of (a), (b) and (c).

4. If \( f(x) = 0 \) for all irrational \( x \), \( f(x) = 1 \) for all rational \( x \), prove that \( f \notin \mathcal{R} \) on \([a,b]\) for any \( a < b \).

**Proof.** Let \( P = \{x_0, x_1, \ldots, x_n\} \) be any partition of \([a,b]\), and \( M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x), m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x), i = 1, \ldots, n \). Since for every \( i \), the interval \([x_{i-1}, x_i]\) contains rationals and irrationals, it follows by the definition of \( f \) that \( M_i = 1, m_i = 0 \). Therefore

\[
U(P, f) = \sum_{i=1}^{n} 1 \times \Delta x_i = b - a, \quad L(P, f) = \sum_{i=1}^{n} 0 \times \Delta x_i = 0.
\]

Since \( P \) is arbitrary, it can then be shown that \( \int f \, dx = b - a > \int f \, dx = 0 \), which means \( f \notin \mathcal{R} \).

5. Suppose \( f \) is a bounded real function on \([a,b]\), and \( f^2 \in \mathcal{R} \) on \([a,b]\). Does it follow that \( f \in \mathcal{R} \)? Does the answer change if we assume that \( f^3 \in \mathcal{R} \)?

**Proof.** Let \( f(x) = 1 \) for all irrational \( x \), \( f(x) = -1 \) for all rational \( x \), then the result of Exercise 4 states that \( f \notin \mathcal{R}(\alpha) \) while \( f^2 \equiv 1 \in \mathcal{R}(\alpha) \).

If \( f^3 \in \mathcal{R}(\alpha) \), since \( f \) is bounded function on \([a,b]\), so is \( f^3 \), assume \( m \leq f^3 \leq M \). Note that \( \phi : [m,M] \to \mathbb{R} \) defined by \( \phi(y) = y^{1/3} \) is continuous on \([m,M]\). Since \( f(x) = \phi(f^3(x)) \), by Theorem 6.11, \( f \in \mathcal{R}(\alpha) \) on \([a,b]\).

6. Let \( P \) be the Cantor set constructed in Sec. 2.44. Let \( f \) be a bounded real function on \([0,1]\) which is continuous at every point outside \( P \). Prove that \( f \in \mathcal{R} \) on \([0,1]\).

**Hint:** \( P \) can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.

**Proof.** Let \( \varepsilon \in (0,1) \) be given. Let \( n \) be chosen such that \( \left( \frac{2}{3} \right)^n < \varepsilon \), which is possible because \( \lim_{n \to \infty} \left( \frac{2}{3} \right)^n = 0 \). For this fixed \( n \), let \( E_n \) be that in Sec. 2.44, which is the
union of $2^n$ (disjoint) intervals, each of length $3^{-n}$. Denote the $2^n$ intervals, from left to right, by $I_{n,k} = [a_{n,k}, b_{n,k}], k = 1, \ldots, 2^n$. For each $k \in \{2, \ldots, 2^n - 1\}$, extending $I_{n,k}$ slightly by setting $a'_{n,k} = a_{n,k} - \frac{1}{9^n}$, $b'_{n,k} = b_{n,k} + \frac{1}{9^n}$, and $J_{n,k} = (a'_{n,k}, b'_{n,k})$.

In addition, let $J_{n,1} = \left[0, \frac{1}{3n} + \frac{1}{9^n}\right] := (a'_{n,1}, b'_{n,1})$, $J_{n,2^n} = \left(1 - \frac{1}{3^n} - \frac{1}{9^n}, 1\right) := (a'_{n,2^n}, b'_{n,2^n})$. It is easily seen that $J_{n,k}, k = 1, \ldots, 2^n$ are disjoint and

$$\sum_{k=1}^{2^n} |J_{n,k}| < \sum_{k=1}^{2^n} \left(\frac{1}{3^n} + \frac{2}{9^n}\right) = \left(\frac{2}{3}\right)^n + 2 \left(\frac{2}{9}\right)^n < \varepsilon + \varepsilon = 2\varepsilon.$$

Remove the segments $J_{n,k}, k = 1, \ldots, 2^n$ from $[0, 1]$. The remaining set $K$ is finite disjoint union of closed intervals, hence is compact. Since $P \subset E_n \subset \bigcup_{i=1}^{2^n} J_{n,k}$, and $f$ is continuous at every point outside $P$, it follows that $f$ is continuous at every point of $K$, hence $f$ is uniformly continuous on $K$, and there exists $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ if $s, t \in K, |s - t| < \delta$.

Now form a partition of $\Pi = \{x_0, x_1, \ldots, x_n\}$ of $[0, 1]$ as follows: Each $a'_{n,k}$ occurs in $\Pi$. Each $b'_{n,k}$ occurs in $\Pi$. No point of any segment $(a'_{n,k}, b'_{n,k})$ occurs in $\Pi$. If $x_{i-1}$ is not one of the $a'_{n,k}$, then $\Delta x_i < \delta$.

Let $M = \sup_{0 \leq x \leq 1} |f(x)|$, note that $M_i - m_i \leq 2M$ for every $i$, and that $M_i - m_i \leq \varepsilon$ unless $x_{i-1}$ is one of the $a'_{n,k}$. Hence

$$U(\Pi, f) - L(\Pi, f) \leq \varepsilon \times (1 - 0) + 2M \times \sum_{k=1}^{2^n} |J_{n,k}| < \varepsilon + 4M\varepsilon.$$

Since $\varepsilon$ is arbitrary, Theorem 6.6 shows that $f \in \mathcal{R}$ on $[0, 1]$.

7. Suppose $f$ is a real function on $(0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x) \, dx = \lim_{c \to 0} \int_c^1 f(x) \, dx$$

if the limit exists (and is finite).

(a) If $f \in \mathcal{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old
one.

(b) Construct a function \( f \) such that the above limit exists, although it fails to exist with \(|f|\) in place of \( f \).

**Proof.** We shall still assume that \( f \) is bounded on \((0, 1]\).

(a) Denote \( F(c) = \int_c^1 f(x) \, dx, c \in (0, 1]\). If \( f \in \mathcal{R} \) on \([0, 1]\) (in which case \( f \) is also defined at 0, and denote \( \sup |f(x)| \) by \( M \)), let \( F(0) = \int_0^1 f(x) \, dx \) be the integral in the old sense. It needs to be shown that \( \lim_{c \to 0} F(c) = F(0) \), i.e., \( F \) is continuous at 0. Indeed, by Theorem 6.12(c):

\[
|F(c) - F(0)| = \left| \int_c^1 f(x) \, dx - \int_0^1 f(x) \, dx \right| = \left| \int_0^c f(x) \, dx \right| \leq Mc \to 0
\]
as \( c \to 0 \). Hence this definition agrees with the old one.

(b) Let \( f(x) = x^{-1} \sin \left( \frac{1}{x} \right), 0 < x \leq 1 \). Notice that, by letting \( t = \frac{1}{x} \),

\[
\lim_{c \to 0} \int_c^1 f(x) \, dx = \lim_{c \to 0} \int_c^1 x^{-1} \sin \left( \frac{1}{x} \right) \, dx = \lim_{c \to 0} \int_c^1 \frac{\sin t}{t} \, dt = \lim_{A \to \infty} \int_1^A \frac{\sin t}{t} \, dt,
\]
so it is equivalent to show that \( \int_1^A \frac{\sin t}{t} \, dt \) exists as \( A \to \infty \). For sufficiently large \( A \), there exists some positive integer \( N \) such that \( A \in ((N - 1)\pi, N\pi] \), write

\[
\int_1^A \frac{\sin t}{t} \, dt = \int_1^\pi \frac{\sin t}{t} \, dt + \sum_{n=2}^{N-1} \int_{(n-1)\pi}^{n\pi} \frac{\sin t}{t} \, dt + \int_{(N-1)\pi}^A \frac{\sin t}{t} \, dt := \sum_{n=1}^{N} c_n.
\]
It is easily seen that \( c_{2n-1} \geq 0, c_{2m} \leq 0, n = 1, 2, 3, \ldots \) and \( \lim_{n \to \infty} c_n = 0 \). In addition, for \( n \geq 1 \),

\[
|c_{2n}| = \left| \int_{n\pi}^{(2n-1)\pi} \frac{\sin t}{t} \, dt \right| = \int_{n\pi}^{(2n-1)\pi} \frac{\sin t}{t} \, dt \geq -\frac{1}{2n\pi} \int_{n\pi}^{(2n-1)\pi} \sin t \, dt = \frac{1}{n\pi},
\]

\[
|c_{2n+1}| = \left| \int_{(2n+1)\pi}^{2n\pi} \frac{\sin t}{t} \, dt \right| = \int_{2n\pi}^{(2n+1)\pi} \frac{\sin t}{t} \, dt \leq \frac{1}{2n\pi} \int_{2n\pi}^{(2n+1)\pi} \sin t \, dt = \frac{1}{n\pi}.
\]
Hence \( |c_{2n}| \geq |c_{2n+1}| \) for every \( n \). Similarly, \( |c_{2n+1}| \geq |c_{2n+2}| \) for every \( n \), hence
we showed that

\[ |c_2| \geq |c_3| \geq |c_4| \geq \cdots. \]

Now the conditions in Theorem 3.43 are all satisfied, hence \( \sum c_n \) converges, consequently, \( \int_1^A \frac{\sin t}{t} \, dt \) exists as \( A \to \infty \).

On the other hand, observe that

\[
\frac{\sin x}{x} \geq \frac{\sin^2 x}{x} = \frac{1}{2x} - \frac{\cos(2x)}{2x}.
\]

In the same manner of showing that \( \int_1^A \sin x/x \, dx \) converges as \( A \to \infty \), it can be shown that \( \int_1^A \cos(2x)/(2x) \, dx \) also converges as \( A \to \infty \). However, since

\[
\int_1^A \frac{1}{x} \, dx = \log A \to \infty
\]

as \( A \to \infty \), we conclude that \( \int_1^A |\sin x|/x \, dx \) diverges as \( A \to \infty \).

8. Suppose \( f \in R \) on \( [a, b] \) for every \( b > a \) where \( a \) is fixed. Define

\[
\int_a^\infty f(x) \, dx = \lim_{b \to \infty} \int_a^b f(x) \, dx
\]

if the limit exists (and is finite). In that case, we say that the integral on the left converges. If it also converges after \( f \) has been replaced by \( |f| \), it is said to converge absolutely.

Assume that \( f(x) \geq 0 \) and that \( f \) decreases monotonically on \([1, \infty)\). Prove that

\[
\int_1^\infty f(x) \, dx
\]

converges if and only if

\[
\sum_{n=1}^{\infty} f(n)
\]

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converges. (This is the so-called “integral test” for convergence of series.)

Proof. If \( \int_{1}^{\infty} f(x) \, dx \) converges, that is, the limit

\[
\lim_{N \to \infty} \int_{1}^{N} f(x) \, dx
\]

exists and is finite. Since \( f \geq 0 \) and is monotonically decreasing, for every positive integer \( N \),

\[
\sum_{n=1}^{N} f(n) = f(1) + f(2) + \cdots + f(N)
\]

\[
\leq f(1) + f(2) \times (2 - 1) + \cdots + f(N) \times (N - (N - 1))
\]

\[
\leq f(1) + \int_{1}^{2} f(x) \, dx + \cdots + \int_{N-1}^{N} f(x) \, dx
\]

\[
= f(1) + \int_{1}^{N} f(x) \, dx
\]

The existence of \( \int_{1}^{N} f(x) \, dx \) as \( N \to \infty \) implies that the partial sum of the series \( \sum f(n) \) converges as \( N \to \infty \), i.e., \( \sum f(n) \) converges.

Conversely, if \( \sum f(n) < \infty \), then for every positive integer \( N \),

\[
\int_{1}^{N} f(x) \, dx = \sum_{n=1}^{N-1} \int_{n}^{n+1} f(x) \, dx \leq \sum_{n=1}^{N-1} f(n).
\]

Therefore \( \int_{1}^{\infty} f(x) \, dx \) converges.

9. Show that integration by parts can sometimes be applied to the “improper” integrals defined in Exercise 7 and 8. (State appropriate hypothesis, formulate a theorem, and prove it.) For instance, show that

\[
\int_{0}^{\infty} \frac{\cos x}{1 + x} \, dx = \int_{0}^{\infty} \frac{\sin x}{(1 + x)^2} \, dx.
\]

Show that one of these integrals converges absolutely, but that the other does not.
Proof. We will only deal with the improper integration of the type in Exercise 8, the other type is similar.

**Theorem (improper integration by parts)** Suppose $F$ and $G$ are differentiable functions on $[a,b]$ for every $b > a$, where $a$ is fixed. For every $b > a$, $F' = f \in \mathbb{R}$ on $[a,b]$, and $G' = g \in \mathbb{R}$ on $[a,b]$. In addition, assume $F(\infty)G(\infty) = \lim_{b \to \infty} F(b)G(b)$ exists, $\int_{a}^{\infty} F(x)g(x)\,dx$ and $\int_{a}^{b} f(x)G(x)\,dx$ converge. Then (in the sense of Exercise 8)

$$\int_{a}^{\infty} F(x)g(x)\,dx = F(\infty)G(\infty) - F(a)G(a) - \int_{a}^{\infty} f(x)G(x)\,dx.$$

**Proof.** By Theorem 6.22, for every $b > a$, we have

$$\int_{a}^{b} F(x)g(x)\,dx = F(b)G(b) - F(a)G(a) - \int_{a}^{b} f(x)G(x)\,dx.$$

The result then follows by passing $b$ to $\infty$ on both sides of the above expression.

Since

$$\int_{0}^{\infty} \frac{|\sin x|}{(1+x)^2} \, dx \leq \int_{0}^{\infty} \frac{1}{(1+x)^2} \, dx = 1 - \lim_{x \to \infty} \frac{1}{1+x} = 1 < \infty,$$

$\int_{0}^{\infty} \frac{\sin x}{(1+x)^2} \, dx$ converges absolutely. On the other hand, since $\int_{1}^{\infty} \sin t/t \, dt$ and $\int_{1}^{\infty} \cos t/t \, dt$ converge (see Exercise 7(b)),

$$\int_{0}^{\infty} \frac{\cos x}{1+x} \, dx = \int_{1}^{\infty} \frac{\cos(t-1)}{t} \, dt = \cos 1 \times \int_{1}^{\infty} \frac{\cos t}{t} \, dt + \sin 1 \times \int_{1}^{\infty} \frac{\sin t}{t} \, dt$$

converges. In addition,

$$\lim_{x \to \infty} \frac{\sin x}{1+x} = 0.$$ 

Hence the conditions of the Theorem stated in this problem are all satisfied, thus we have

$$\int_{0}^{\infty} \frac{\cos x}{1+x} \, dx = \lim_{x \to \infty} \frac{\sin x}{1+x} - \frac{\sin 0}{1+0} + \int_{0}^{\infty} \frac{\sin x}{(1+x)^2} \, dx = \int_{0}^{\infty} \frac{\sin x}{(1+x)^2} \, dx.$$

It remains to show that $\int_{0}^{\infty} \frac{\cos x}{1+x} \, dx$ does not converge absolutely, which can be shown
in the same manner as that in Exercise 7(b) (use the inequality \(|\cos(t - 1)| \geq \cos^2(t - 1)|\)).

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10. Let \( p \) and \( q \) be positive real numbers such that

\[ \frac{1}{p} + \frac{1}{q} = 1. \]

Prove the following statements.

(a) If \( u \geq 0 \) and \( v \geq 0 \), then

\[ uv \leq \frac{u^p}{p} + \frac{v^q}{q}. \]

Equality holds if and only if \( u^p = v^q \).

(b) If \( f \in \mathcal{R}(\alpha), g \in \mathcal{R}(\alpha), f \geq 0, g \geq 0, \) and

\[ \int_a^b f^p \, d\alpha = 1 = \int_a^b g^q \, d\alpha, \]

then

\[ \int_a^b fg \, d\alpha \leq 1. \]

(c) If \( f \) and \( g \) are complex functions in \( \mathcal{R}(\alpha) \), then

\[ \left| \int_a^b fg \, d\alpha \right| \leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, d\alpha \right\}^{1/q}. \]

This is Hölder’s inequality. When \( p = q = 2 \) it is usually called Schwarz inequality,

(Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder’s inequality is also true for the “improper” integrals described in Exercises 7 and 8.

Proof.

(a) If either \( u = 0 \) or \( v = 0 \) there is nothing to prove. So we may assume \( u > 0, v > 0, \)

for which there exist \( s \in \mathbb{R}, t \in \mathbb{R}, \) such that \( u = e^{\frac{1}{p}s}, v = e^{\frac{1}{q}t}. \) Since the function
$e^x$ is strictly convex on $\mathbb{R}^1$, it follows that

$$uv = \exp \left( \frac{1}{p} s + \frac{1}{q} t \right) \leq \frac{1}{p} e^s + \frac{1}{q} e^t = \frac{1}{p} u^p + \frac{1}{q} v^q.$$ 

The equality holds if and only if $s = t$, i.e., $u^p = v^q$.

(b) Since $f \in \mathcal{R}(\alpha), g \in \mathcal{R}(\alpha)$, by Theorem 6.13, $fg \in \mathcal{R}(\alpha)$. Given $\varepsilon > 0$, since $\int_a^b f^p \, d\alpha = \int_a^b g^q \, d\alpha = 1$, by Theorem 6.7(c) (the concept of refinement partition are used implicitly here), there exists some partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$\left| \sum_{i=1}^n f(x_i)^p \Delta \alpha_i - 1 \right| < \varepsilon, \quad \left| \sum_{i=1}^n g(x_i)^q \Delta \alpha_i - 1 \right| < \varepsilon,$$

$$\left| \sum_{i=1}^n f(x_i)g(x_i) \Delta \alpha_i - \int_a^b fg \, d\alpha \right| < \varepsilon.$$

On the other hand, by the inequality proved in part (a) (which known as “Young’s inequality”):

$$\sum_{i=1}^n f(x_i)g(x_i) \Delta \alpha_i - 1$$

$$\leq \sum_{i=1}^n \left( \frac{f(x_i)^p}{p} + \frac{g(x_i)^q}{q} \right) \Delta \alpha_i - 1$$

$$= \frac{1}{p} \left( \sum_{i=1}^n f(x_i)^p \Delta \alpha_i - 1 \right) + \frac{1}{q} \left( \sum_{i=1}^n g(x_i)^q \Delta \alpha_i - 1 \right)$$

$$< \frac{1}{p} \varepsilon + \frac{1}{q} \varepsilon = \varepsilon.$$

Therefore,

$$\int_a^b fg \, d\alpha < \sum_{i=1}^n f(x_i)g(x_i) \Delta \alpha_i + \varepsilon < 1 + \varepsilon + \varepsilon = 1 + 2\varepsilon.$$

Since $\varepsilon$ is arbitrarily small, it follows that $\int_a^b fg \, d\alpha \leq 1$.

(c) If either $\int_a^b |f|^p \, d\alpha = 0$ or $\int_a^b |g|^q \, d\alpha = 0$, it can be shown by definition that $f = 0$
or \( g = 0 \) almost everywhere on \([a, b]\), hence the inequality holds. Otherwise, let
\[
\tilde{f} = \frac{|f|}{\left\{ \int_a^b |f|^p \, d\alpha \right\}^{\frac{1}{p}}}, \quad \tilde{g} = \frac{|g|}{\left\{ \int_a^b |g|^q \, d\alpha \right\}^{\frac{1}{q}}}.
\]

Clearly, the conditions in part (b) are all satisfied by real functions \( \tilde{f} \) and \( \tilde{g} \), hence
\[
\int_a^b \tilde{f} \tilde{g} \, d\alpha \leq 1.
\]
Consequently,
\[
\left| \int_a^b fg \, d\alpha \right| \leq \int_a^b |fg| \, d\alpha = \left\{ \int_a^b |f|^p \, d\alpha \right\}^{\frac{1}{p}} \cdot \left\{ \int_a^b |g|^q \, d\alpha \right\}^{\frac{1}{q}} \cdot \int_a^b \tilde{f} \tilde{g} \, d\alpha
\]
\[
\leq \left\{ \int_a^b |f|^p \, d\alpha \right\}^{\frac{1}{p}} \cdot \left\{ \int_a^b |g|^q \, d\alpha \right\}^{\frac{1}{q}} \cdot 1 = \left\{ \int_a^b |f|^p \, d\alpha \right\}^{\frac{1}{p}} \cdot \left\{ \int_a^b |g|^q \, d\alpha \right\}^{\frac{1}{q}} \cdot \left( \int_a^b \tilde{f} \tilde{g} \, d\alpha \right).
\]

(d) For the improper integral in Exercise 8, let \( b \to \infty \) on both sides of the inequality proved in part (d), by the continuity of the indefinite integral, the inequality still holds. For the improper integral in Exercise 7, the treatment is similar.

11. Let \( \alpha \) be a fixed increasing function on \([a, b]\). For \( u \in \mathcal{R}(\alpha) \), define
\[
\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.
\]
Suppose \( f, g, h \in \mathcal{R}(\alpha) \), and prove the triangle inequality
\[
\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2
\]
as a consequence of the Schwarz inequality.
Proof. The inequality is equivalent to
\[
\int_a^b |f-h|^2 \, d\alpha \leq \int_a^b |f-g|^2 \, d\alpha + \int_a^b |g-h|^2 \, d\alpha + 2 \left( \int_a^b |f-g|^2 \, d\alpha \right)^{1/2} \left( \int_a^b |g-h|^2 \, d\alpha \right)^{1/2}.
\]
By the triangle inequality, the left hand side of the above expression is not greater than
\[
\int_a^b (|f-g| + |g-h|)^2 \, d\alpha = \int_a^b |f-g|^2 \, d\alpha + \int_a^b |g-h|^2 \, d\alpha + 2 \int_a^b |f-g||g-h| \, d\alpha.
\]
By Schwarz inequality,
\[
\int_a^b |f-g||g-h| \, d\alpha \leq \left( \int_a^b |f-g|^2 \, d\alpha \right)^{1/2} \left( \int_a^b |g-h|^2 \, d\alpha \right)^{1/2},
\]
hence the result follows.

12. With the notation of Exercise 11, suppose \( f \in \mathcal{D}(\alpha) \) and \( \varepsilon > 0 \). Prove that there exists a continuous function \( g \) on \([a, b]\) such that \( \|f - g\|_2 < \varepsilon \).

**Hint:** Let \( P = \{x_0, \ldots, x_n\} \) be a suitable partition of \([a, b]\), define
\[
g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)
\]
if \( x_{i-1} \leq t \leq x_i \).

**Proof.** Assume \( |f| \leq M \) on \([a, b]\) for some \( M > 0 \). Given \( \varepsilon > 0 \), let \( P = \{x_0, \ldots, x_n\} \) be a partition of \([a, b]\) such that \( U(P, f, \alpha) - L(P, f, \alpha) < \frac{\varepsilon^2}{8M} \). Now let \( g \) be defined as in the hint, clearly, \( g \) is continuous on \([a, b]\). In addition to \( g \), define a step function \( h \) on \([a, b]\) as follows:
\[
h(x) = f(x_{i-1}), \ x \in [x_{i-1}, x_i], \ i = 1, 2, \ldots, n.
\]
We can first evaluate the quadratic discrepancy between \( g \) and \( h \) as follows:

\[
\|g - h\|_2^2 = \int_a^b |g - h|^2 \, d\alpha = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |g - h|^2 \, d\alpha
\]

\[
= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left| \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i) - f(x_{i-1}) \right|^2 \, d\alpha(t)
\]

\[
= \sum_{i=1}^{n} \frac{(f(x_i) - f(x_{i-1}))^2}{(\Delta x_i)^2} \int_{x_{i-1}}^{x_i} (t - x_{i-1})^2 \, d\alpha(t)
\]

\[
\leq \sum_{i=1}^{n} \frac{(f(x_i) - f(x_{i-1}))^2}{(\Delta x_i)^2} (x_i - x_{i-1})^2 (\alpha(x_i) - \alpha(x_{i-1}))
\]

\[
= \sum_{i=1}^{n} ((f(x_i) - f(x_{i-1}))^2) (\alpha(x_i) - \alpha(x_{i-1}))
\]

\[
\leq 2M \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \Delta \alpha_i
\]

\[
\leq 2M(U(P, f, \alpha) - L(P, f, \alpha)) < \frac{\varepsilon^2}{4}.
\]

Next, it can be shown that

\[
\|f - h\|_2^2 = \int_a^b |f - h|^2 \, d\alpha = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f - h|^2 \, d\alpha
\]

\[
= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t) - f(x_{i-1})|^2 \, d\alpha(t)
\]

\[
\leq 2M \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t) - f(x_{i-1})| \, d\alpha(t)
\]

\[
\leq 2M \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (M_i - m_i) \, d\alpha(t)
\]

\[
= 2M \sum_{i=1}^{n} (M_i - m_i) \Delta \alpha_i
\]

\[
= 2M(U(P, f, \alpha) - L(P, f, \alpha)) < \frac{\varepsilon^2}{4}.
\]

Therefore, by the inequality proved in Exercise 13, we have

\[
\|f - g\|_2 \leq \|f - h\|_2 + \|g - h\|_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
13. Define

\[ f(x) = \int_x^{x+1} \sin(t^2) \, dt. \]

(a) Prove that \(|f(x)| < 1/x\) if \(x > 0\).

(b) Prove that

\[ 2xf(x) = \cos(x^2) - \cos[(x + 1)^2] + r(x) \]

where \(|r(x)| < c/x\) and \(c\) is a constant.

(c) Find the upper and lower limits of \(xf(x)\), as \(x \to \infty\).

(d) Does \(\int_0^\infty \sin(t^2) \, dt\) converge?

**Proof.**

(a) By hint, using integration by parts, we obtain

\[ f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos[(x + 1)^2]}{2(x + 1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} \, du. \]

So if \(x > 0\),

\[ |f(x)| < \frac{1}{2x} + \frac{1}{2(x + 1)} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} \, du = \frac{1}{x}. \]

Note the inequality is strict for \(|\cos u|\) doesn’t equal to 1 at every point of \([x^2, (x + 1)^2]\).

(b) By part (a)

\[ 2xf(x) = \cos(x^2) - \frac{x}{x + 1} \cos[(x + 1)^2] - 2x \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} \, du \]

\[ = \cos(x^2) - \cos[(x + 1)^2] + \frac{1}{x + 1} \cos[(x + 1)^2] - 2x \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} \, du. \]
Therefore,

\[ |r(x)| = \frac{1}{x+1} \cos((x+1)^2) - 2x \int_{x^2}^{(x+1)^2} \cos \frac{u}{4u^{3/2}} \, du \]
\[ \leq \frac{1}{x+1} + 2x \left[ -\frac{1}{2} \left( \frac{1}{x+1} - \frac{1}{x} \right) \right] \]
\[ = \frac{2}{x+1} < \frac{2}{x}. \]

Let \( g(x) = \cos(x^2) - \cos((x+1)^2), \) \( x \in \mathbb{R}, \) we will show that

\[ \limsup_{x \to \infty} g(x) = 2, \quad \liminf_{x \to \infty} g(x) = -2. \]

Consider the sequence \( \{x_n\} = \{2n\sqrt{\pi}\} \) which tends to \( \infty \) as \( n \to \infty. \) It is easily seen that

\[ g(x_n) = \cos(4n^2\pi) - \cos(4n^2\pi + 4n\sqrt{\pi} + 1) = 1 - \cos(n\alpha + 1) \]

where \( \alpha := 4\sqrt{\pi}. \) Since \( \frac{\alpha}{2\pi} = \frac{2}{\sqrt{\pi}} > 0 \) is an irrational number, with the same argument as Chap. 4, Exercise 25(b), it can be shown that the set

\[ \left\{ n\alpha - 2\pi \left\lfloor \frac{n\alpha}{2\pi} \right\rfloor : n \in \mathbb{N} \right\} \]

is dense in \([0, 2\pi].\) Two facts that are useful in what follows:

(i) If \( m \neq n, \) then \( m\alpha - 2\pi \left\lfloor \frac{m\alpha}{2\pi} \right\rfloor \neq n\alpha - 2\pi \left\lfloor \frac{n\alpha}{2\pi} \right\rfloor; \)

(ii) For every \( n \in \mathbb{N}, \)

\[ n\alpha - 2\pi \left\lfloor \frac{n\alpha}{2\pi} \right\rfloor \neq \pi - 1. \]

(otherwise, \( \pi \) would be an algebraic number, which is false.)

For \( n = 1, \) let

\[ k_1 = \inf \left\{ k \geq 1 : \left| k\alpha - 2\pi \left\lfloor \frac{k\alpha}{2\pi} \right\rfloor - (\pi - 1) \right| < 1 \right\}. \]

---

1Rigorous proof writing is hard!
$k_1$ must exist since the aforementioned set is dense in $[0, 2\pi]$ and $\pi - 1 \in [0, 2\pi]$.

Denote $d_1 = |k_1\alpha - 2\pi\left\lfloor \frac{k_1\alpha}{2\pi} \right\rfloor - (\pi - 1)|$. Because of (ii), $d_1 > 0$.

For $n = 2$, let

$$k_2 = \inf\left\{ k > k_1 : \left| k\alpha - 2\pi\left\lfloor \frac{k\alpha}{2\pi} \right\rfloor - (\pi - 1) \right| < \min\left(\frac{1}{2}, d_1\right) \right\}.$$ 

Then define $d_2$ similarly.

Recursively, for each $n$, define

$$k_n = \inf\left\{ k > k_{n-1} : \left| k\alpha - 2\pi\left\lfloor \frac{k\alpha}{2\pi} \right\rfloor - (\pi - 1) \right| < \min\left(\frac{1}{n}, d_{n-1}\right) \right\}.$$ 

In this way we obtain a subsequence $\{k_n\}$ that tends to infinity.

Since for any $x \in \mathbb{R}^1$, $y \in \mathbb{R}^1$, $|\cos x - \cos y| \leq |x - y|$, it follows that

$$\left| \cos(k_n\alpha + 1) - (-1) \right| = \left| \cos(k_n\alpha + 1) - \cos(2\pi\left\lfloor (k_n\alpha)/(2\pi) \right\rfloor + \pi) \right|$$

$$\leq \left| k_n\alpha + 1 - 2\pi\left\lfloor \frac{k_n\alpha}{2\pi} \right\rfloor - \pi \right|$$

$$= \left| k_n\alpha - 2\pi\left\lfloor \frac{k_n\alpha}{2\pi} \right\rfloor - (\pi - 1) \right|$$

$$< \frac{1}{n}$$

Therefore, $\lim_{n \to \infty} \cos(k_n\alpha + 1) = -1$. Consequently,

$$\lim_{n \to \infty} g(x_{k_n}) = 1 - \lim_{n \to \infty} \cos(k_n\alpha + 1) = 2.$$ 

On the other hand, since $g(x) \leq 2$ for all real $x$, it follows that $\limsup_{x \to \infty} g(x) = 2$.

It can be shown that $\liminf_{x \to \infty} g(x) = -2$ in a similar manner.

By the identity showed in part (b), we have

$$\limsup_{x \to \infty} xf(x) = 1, \quad \liminf_{x \to \infty} xf(x) = -1.$$
(d) Clearly, $\sin(t^2)$ is integrable on $[0, 1]$. For every $A > 1$, integrating by parts gives

$$\int_1^A \sin(t^2) \, dt = \frac{\cos 1}{2} - \frac{\cos(A^2)}{2A} - \int_1^A \frac{\cos u}{4u^{3/2}} \, du.$$ 

In the same way as in Exercise 7, part (b), it can be shown that $\int_1^A \frac{\cos u}{4u^{3/2}} \, du$ converges as $A \to \infty$. In addition to $\lim_{A \to \infty} \frac{\cos(A^2)}{2A} = 0$, we conclude that $\int_0^\infty \sin(t^2) \, dt$ converges.

\[\square\]

14. Deal similarly with

$$f(x) = \int_x^{x+1} \sin(e^t) \, dt.$$ 

Show that

$$e^x |f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) + r(x),$$

where $|r(x)| < Ce^{-x}$, for some constant $C$.

**Proof.** Using integration by parts, it can be shown that

$$e^x f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) - e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} \, du.$$ 

Thus $r(x) = e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} \, du$, and

$$|r(x)| \leq e^x \int \frac{1}{u^2} \, du = 1 - e^{-1}.$$ 

Therefore

$$|e^x f(x)| \leq 1 + e^{-1} + 1 - e^{-1} = 2.$$  

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To get a sharper upper bound for $|r(x)|$, we need to do another integration by parts:

$$
\int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} \, du = \frac{\sin(e^{x+1})}{e^{2(x+1)}} - \frac{\sin(e^x)}{e^{2x}} + 2 \int_{e^x}^{e^{x+1}} \frac{\sin u}{u^3} \, du
$$

Therefore

$$
e^x |r(x)| = \left| \frac{\sin(e^{x+1})}{e^2} - \sin(e^x) + 2e^{2x} \int_{e^x}^{e^{x+1}} \frac{\sin u}{u^3} \, du \right|
$$

$$
< \frac{1}{e^2} + 1 + 2e^{2x} \int_{e^x}^{e^{x+1}} \frac{1}{u^3} \, du
$$

$$
= \frac{1}{e^2} + 1 + e^{2x} (e^{-2x} - e^{-2(x+1)})
$$

$$
= \frac{1}{e^2} + 1 + 1 - \frac{1}{e^2}
$$

$$
= 2.
$$

\[\square\]

15. Suppose $f$ is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$
\int_a^b f^2(x) \, dx = 1.
$$

Prove that

$$
\int_a^b xf(x)f'(x) \, dx = -\frac{1}{2}
$$

and that

$$
\int_a^b [f'(x)]^2 \, dx \cdot \int_a^b x^2 f^2(x) \, dx > \frac{1}{2}.
$$

Proof. Integrating by parts, it follows that

$$
\int_a^b xf(x)f'(x) \, dx = xf^2(x) \big|_a^b - \int_a^b f(x)f(x) + xf'(x) \, dx
$$

$$
= 0 - \int_a^b f^2(x) \, dx - \int_a^b xf(x)f'(x) \, dx
$$

$$
= -1 - \int_a^b xf(x)f'(x) \, dx,
$$

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Hence
\[ \int_a^b x f(x) f'(x) \, dx = -\frac{1}{2}. \]

By Cauchy-Schwarz inequality,
\[ \frac{1}{4} = \left( \int_a^b x f(x) \cdot f'(x) \, dx \right)^2 \leq \int_a^b x^2 f^2(x) \, dx \cdot \int_a^b [f'(x)]^2 \, dx. \]

Suppose
\[ \int_a^b [f'(x)]^2 \, dx \cdot \int_a^b x^2 f^2(x) \, dx = \frac{1}{4}, \]
then there exists some constant \( \lambda \) such that \( f'(x) = \lambda x f(x) \) for all \( x \in [a,b] \). We shall show under this case \( f \) is identically zero on \( [a,b] \), which is a contradiction to the condition \( \int_a^b f^2(x) \, dx = 0. \)

If there exists \( x_0 \in (a,b) \) such that \( f(x_0) \neq 0 \). Let
\[ b = \inf\{ x \in (a,x_0) : f(t) \neq 0 \text{ for all } t \in (x,x_0) \}. \]

Note \( a \leq b < x_0 \) by continuity of \( f \). In addition \( f(b) = \lim_{t \uparrow b} f(t) = 0. \)

In the segment \( (b,x_0) \), we have
\[ \frac{d \log(|f(x)|)}{dx} = \frac{f'(x)}{f(x)} = \lambda x. \]

and hence \( f(x) = ce^{\frac{1}{2} \lambda x^2}, x \in (b,x_0) \). By continuity of \( f \) at \( b \), it follows that \( c = 0. \)

Again by continuity of \( f \) at \( x_0 \), \( f(x_0) = 0 \), which is a contradiction.

\[ \square \]

16. For \( 1 < s < \infty \), define
\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \]

(This is \textit{Riemann's zeta function}, of great importance in the study of the distribution of prime numbers.) Prove that
(a) \( \zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}} \, dx \) and that
(b) \( \zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{x-[x]}{x^{s+1}} \, dx \),
where \([x]\) denotes the greatest integer \( \leq x \).

Prove that the integral in (b) converges for all \( s > 0 \).

\textit{Hint:} To prove (a), compute the difference between the integer over \([1,N]\) and the \(N\)th partial sum of the series that defines \( \zeta(s) \).

\textbf{Proof.}

(a) First, since \([x] \leq x\) for all \( x > 1 \),
\[
0 < \int_1^\infty \frac{[x]}{x^{s+1}} \, dx \leq \int_1^\infty \frac{1}{x^s} \, dx < \infty.
\]
i.e., the improper integration converges when \( s > 1 \). To show the equality, for every positive integer \( N > 1 \), by the summation by parts formula (Theorem 3.41),
\[
\sum_{n=1}^{N-1} n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) = \sum_{n=1}^{N-1} \frac{1}{n^s} - \frac{N-1}{N^s}.
\]
Therefore,
\[
\left| \sum_{n=1}^{N} \frac{1}{n^s} - s \int_1^N \frac{[x]}{x^{s+1}} \, dx \right| = \left| \sum_{n=1}^{N} \frac{1}{n^s} - s \sum_{n=1}^{N-1} \int_n^{n+1} \frac{[x]}{x^{s+1}} \, dx \right|
\]
\[
= \left| \sum_{n=1}^{N} \frac{1}{n^s} - s \sum_{n=1}^{N-1} n \int_n^{n+1} \frac{1}{x^{s+1}} \, dx \right|
\]
\[
= \left| \sum_{n=1}^{N} \frac{1}{n^s} - \sum_{n=1}^{N-1} n \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right) \right|
\]
\[
= \left| \sum_{n=1}^{N} \frac{1}{n^s} - \left[ \sum_{n=1}^{N-1} \frac{1}{n^s} - \frac{N-1}{N^s} \right] \right|
\]
\[
= \frac{1}{N^s} + \frac{N-1}{N^s} = \frac{1}{N^{s-1}} \to 0
\]
as $N \to \infty$, which proves

$$\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}} \, dx.$$  

(b) For every $N > 1$, if $s > 1$,  

$$0 < \int_1^N \frac{x-[x]}{x^{s+1}} \, dx = \sum_{n=1}^{N-1} \int_n^{n+1} \frac{x-[x]}{x^{s+1}} \, dx$$

$$= \sum_{n=1}^{N-1} \left[ \int_n^{n+1} \frac{1}{x^s} \, dx - n \int_n^{n+1} \frac{1}{x^{s+1}} \, dx \right]$$

$$= \sum_{n=1}^{N-1} \left\{ \frac{1}{s-1} \left[ \frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}} \right] - \frac{n}{s} \left[ \frac{1}{n^s} - \frac{1}{(n+1)^s} \right] \right\}$$

$$< \sum_{n=1}^{N-1} \frac{1}{s(s-1)} \left[ \frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}} \right] \quad \text{where we used} \quad \frac{n}{n+1} < 1.$$  

$$= \frac{1}{s(s-1)} \left( 1 - \frac{1}{N^{s-1}} \right) \to \frac{1}{s(s-1)}$$

as $N \to \infty$. So the integral converges for all $s > 1$.

If $s = 1$, then for every $N > 1$,  

$$0 < \int_1^N \frac{x-[x]}{x^2} \, dx = \sum_{n=1}^{N-1} \int_n^{n+1} \frac{x-[x]}{x^2} \, dx$$

$$= \sum_{n=1}^{N-1} \left[ \int_n^{n+1} \frac{1}{x} \, dx - n \int_n^{n+1} \frac{1}{x^2} \, dx \right]$$

$$= \sum_{n=1}^{N-1} \left\{ \left[ \log(n+1) - \log n \right] - n \left[ \frac{1}{n} - \frac{1}{n+1} \right] \right\}$$

$$= \log N - \sum_{n=2}^{N} \frac{1}{n}$$

$$= \log N - (\log N + \gamma + \varepsilon_N - 1)$$

$$= 1 - \gamma - \varepsilon_N \to 1 - \gamma$$
as $N \to \infty$, where $\gamma$ is Euler constant and $\varepsilon_N = o(1)$ as $N \to \infty$. So the integral converges for $s = 1$.

If $0 < s < 1$, then for every $N > 1$,

$$0 < \int_1^N \frac{x-[x]}{x^{s+1}} \, dx = \sum_{n=1}^{N-1} \int_n^{n+1} \frac{x-[x]}{x^{s+1}} \, dx$$

$$= \sum_{n=1}^{N-1} \left[ \int_n^{n+1} \frac{1}{x^s} \, dx - n \int_n^{n+1} \frac{1}{x^{s+1}} \, dx \right]$$

$$= \sum_{n=1}^{N-1} \left\{ \frac{1}{1-s} \left[ (n+1)^{1-s} - n^{1-s} \right] - \frac{1}{s} \left[ n^{1-s} - \frac{n}{(n+1)^s} \right] \right\}$$

$$= \sum_{n=1}^{N-1} \left\{ \frac{1}{1-s} \left[ (n+1)^{1-s} - n^{1-s} \right] - \frac{1}{s} \left[ n^{1-s} - (n+1)^{1-s} + \frac{1}{(n+1)^s} \right] \right\}$$

$$= \frac{1}{s(1-s)} (N^{1-s} - 1) - \frac{1}{s} \sum_{n=1}^{N-1} \frac{1}{(n+1)^s}$$

$$\to \frac{1}{1-s} (2^{1-s} - 1)$$

as $N \to \infty$. At the last step, we used the integration estimation of the partial sum:

$$\sum_{n=1}^{N-1} \frac{1}{(n+1)^s} = \int_2^N \frac{1}{x^s} \, dx + O \left( \frac{1}{N^s} \right).$$

So the integral converges for all $s \in (0,1)$.

Now for $s > 1$ and $N > 2$,

$$\left| \sum_{n=1}^N \frac{1}{n^s} - \left( \frac{s}{1-s} - s \int_1^N \frac{x-[x]}{x^{s+1}} \, dx \right) \right|$$

$$= \left| \sum_{n=2}^N \frac{1}{n^s} + \sum_{n=2}^{N-1} \left\{ \frac{1}{s-1} \frac{1}{n^{s-1}} - \frac{1}{s-1} \frac{n + s}{(n+1)^s} \right\} \right|$$

$$= \left| \sum_{n=2}^{N-1} \frac{1}{n^{s+1}} - \sum_{n=2}^{N-1} \frac{1}{n^{s}} - \sum_{n=1}^{N-1} \frac{1}{(n+1)^s} - \sum_{n=1}^{N-1} \frac{1}{(n+1)^{s-1}} \right|$$

$$= \frac{1}{(s-1)N^{s-1}} \to 0$$

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as $N \to \infty$, which shows

$$\zeta(s) = \frac{s}{s-1} - s \int_{0}^{\infty} \frac{x - [x]}{x^{s+1}} \, dx.$$ 

17. Suppose $\alpha$ increases monotonically on $[a, b]$, $g$ is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_{a}^{b} \alpha(x)g(x) \, dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_{a}^{b} G \, d\alpha.$$ 

Hint: Take $g$ real, without loss of generality. Give $P = \{x_0, x_1, \ldots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta \alpha_i.$$ 

Proof. Let $\varepsilon > 0$ be given. Since $g$ is continuous on $[a, b]$, $g$ is uniformly continuous. Hence there exists $\delta > 0$ such that whenever $s \in [a, b], t \in [a, b], |s - t| < \delta, |g(s) - g(t)| < \varepsilon$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i < \delta$ for all $i$, and such that

$$\left| \sum_{i=1}^{n} \alpha(x_i)g(x_i)\Delta x_i - \int_{a}^{b} \alpha(x)g(x) \, dx \right| < \varepsilon, \quad \left| \sum_{i=1}^{n} G(x_{i-1})\Delta \alpha_i - \int_{a}^{b} G \, d\alpha \right| < \varepsilon.$$ 

By the mean value theorem, for each $i$, there exists $t_i \in (x_{i-1}, x_i)$ such that $G(x_i) - G(x_{i-1}) = g(t_i)\Delta x_i$. Therefore, by the summation by parts formula,

$$\sum_{i=1}^{n} \alpha(x_i)g(t_i)\Delta x_i = \sum_{i=1}^{n} \alpha(x_i)(G(x_i) - G(x_{i-1})) = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta \alpha_i.$$ 

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Assume $|\alpha(x)| \leq M$ for all $x \in [a, b]$, it then follows that

$$
\left| \int_{a}^{b} \alpha(x) g(x) \, dx - \left( G(b)\alpha(b) - G(a)\alpha(a) - \int_{a}^{b} G \, d\alpha \right) \right| \\
\leq \int_{a}^{b} |\alpha(x)| g(x) \, dx + \int_{a}^{b} \left| \alpha(x) g(x) \right| dx \\
+ \left| \sum_{i=1}^{n} \alpha(x_i) g(x_i) \Delta x_i \right| \left( G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1}) \Delta \alpha_i \right) \\
+ \left| \sum_{i=1}^{n} G(x_{i-1}) \Delta \alpha_i - \int_{a}^{b} G \, d\alpha \right| \\
< \varepsilon + \sum_{i=1}^{n} |\alpha(x_i)||g(x_i) - g(t_i)| \Delta x_i + 0 + \varepsilon \\
< 2\varepsilon + M(b-a)\varepsilon = [M(b-a) + 2]\varepsilon.
$$

Since $\varepsilon$ was arbitrarily small, the result follows. \(\square\)

18. Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane, defined on $[0, 2\pi]$ by

$$
\gamma_1(t) = e^{it}, \quad \gamma_2(t) = e^{2it}, \quad \gamma_3(t) = e^{2\pi it \sin(1/t)}.
$$

Show that these three curves have the same range, that $\gamma_1$ and $\gamma_2$ are rectifiable, that the length of $\gamma_1$ is $2\pi$, that the length of $\gamma_2$ is $4\pi$, and that $\gamma_3$ is not rectifiable.

**Proof.** It is clear that both $\gamma_1$ and $\gamma_2$ have the ranges of the unit circle $S = \{z \in \mathbb{C} : |z| = 1\}$. To show that $\gamma_3$’s range is also $S$, it suffices to show the range of the function $g(t) = 2\pi t \sin(1/t), t \in [0, 2\pi]$ contains $[0, 2\pi]$. It is easy to check that

$$
g(0) = 0, \quad g \left( \frac{2}{\pi} \right) = 2\pi \times \frac{\pi}{2} \times \sin \left( \frac{\pi}{2} \right) = \pi^2 > 2\pi.
$$

Since $g$ is continuous on $[0, 2/\pi]$, by intermediate theorem (Theorem 4.22), $[0, 2\pi] \subset g([0, 2/\pi]) \subset g([0, 2\pi])$. Thus the range of $\gamma_3$ is also $S$.

Since

$$
\gamma'_1(t) = ie^{it}, \quad \gamma'_2(t) = 2ie^{2it}
$$

which are continuous on $[0, 2\pi]$, by Theorem 6.27, $\gamma_1, \gamma_2$ are rectifiable. In addition,
by the same theorem,

\[ \Lambda(\gamma_1) = \int_0^{2\pi} |ie^{it}| \, dt = 2\pi, \quad \Lambda(\gamma_2) = \int_0^{2\pi} |2ie^{it}| \, dt = 4\pi. \]

To show \( \gamma_3 \) is not rectifiable, we need to show \( \Lambda(\gamma_3) = \infty \). For each positive integer \( n \), let

\[ P_n = \left\{ 0, \frac{1}{2n\pi} + \frac{\pi}{2}, \frac{1}{2n\pi}, \ldots, \frac{1}{2n\pi} + \frac{\pi}{2}, \frac{1}{2n\pi} \right\} \]

denote the partition of \([0, 2\pi]\). Since for every \( k \in \{1, \ldots, n\} \)

\[ \left| \gamma_3 \left( \frac{1}{2k\pi} \right) - \gamma_3 \left( \frac{1}{2k\pi} + \frac{\pi}{2} \right) \right| = 2 \sin \left( \frac{2}{4n + 1} \right) = \frac{2}{4k + 1} + O \left( \frac{1}{(4k + 1)^3} \right), \]

it follows that

\[ \Lambda(\gamma_3, P_n) > \sum_{k=1}^{n} \left| \gamma_3 \left( \frac{1}{2k\pi} \right) - \gamma_3 \left( \frac{1}{2k\pi} + \frac{\pi}{2} \right) \right| = \sum_{k=1}^{n} \frac{2}{4k + 1} + O \left( \sum_{k=1}^{n} \frac{1}{(4k + 1)^3} \right). \]

Therefore, because of the series \( \sum \frac{2}{4k+1} \) diverges, and \( \sum \frac{1}{(4k+1)^3} \) converges,

\[ \Lambda(\gamma_3) \geq \sup_n \Lambda(\gamma_3, P_n) = \infty. \]

i.e., the curve \( \gamma_3 \) is not rectifiable.

19. Let \( \gamma_1 \) be a curve in \( \mathbb{R}^k \), defined on \( [a, b] \); let \( \phi \) be a continuous 1-1 mapping of \( [c, d] \) onto \([a, b]\), such that \( \phi(c) = a \); and define \( \gamma_2(s) = \gamma_1(\phi(s)) \). Prove that \( \gamma_2 \) is an arc, a closed curve, or a rectifiable curve if and only if the same is true of \( \gamma_1 \). Prove that \( \gamma_2 \) and \( \gamma_1 \) have the same length.
Proof. Because $\phi$ is one-to-one mapping of $[c, d]$ onto $[a, b]$, $\gamma_2$ is one-to-one if and only if $\gamma_1$ is one-to-one, in other words, $\gamma_2$ is an arc if and only if $\gamma_1$ is an arc.

We first show that it necessarily holds that $\phi(d) = b$. Otherwise, there exists $x_0 \in (c, d)$ such that $\phi(x_0) = b$ and $e \in (a, b)$ such that $\phi(d) = e$. It then follows by the intermediate theorem that there exists $y_0 \in (c, x_0)$ such that $\phi(y_0) = e$, contradicts with that $\phi$ is one-to-one. Therefore if $\gamma_1(a) = \gamma_1(b)$, then $\gamma_2(c) = \gamma_1(\phi(c)) = \gamma_1(a) = \gamma_1(b) = \gamma_2(\phi(d)) = \gamma_2(d)$ and vice versa.

Finally, let $\varphi$ be the inverse mapping of $\phi$, which maps $[a, b]$ onto $[c, d]$. Since $\phi$ and $\varphi$ establish a one-to-one correspondence between partitions $\{s_i\}$ of $[a, b]$ and $\{t_i\}$ of $[c, d]$ such that

$$\sum |\gamma_1(s_i) - \gamma_1(s_{i-1})| = \sum |\gamma_2(t_i - \gamma_2(t_{i-1})|,$$

it follows that the two curves have the same length (which implies that $\gamma_1$ and $\gamma_2$ are simultaneously rectifiable or not rectifiable.

$\hfill \Box$