Research Statement

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1 Overview

My main interests lie in the field of applied mathematics, specifically, in modeling and controlling physical phenomena as well as in numerical analysis of Partial Differential Equations (PDEs) that describe physical processes. I am particularly intrigued by how mathematics builds a bridge between physics and engineering through the use of analytical and computational techniques.

In what follows, I will outline my main results and proposed future research. In Sect. 2, I present my work on inhomogeneous media. An overview of a numerical method for solving eigenproblems is provided in Sect. 3. Sect. 4 summarizes my papers on controlling vibrating and heat conducting systems. In Sect. 5, I discuss an application of diffraction theory. Finally, in Sect. 6, I conclude this statement with my research plans.

2 Mechanics of Inhomogeneous Continua

2.1 Spectral problems: explaining vibrations of composite media.

While classical homogeneous media, e.g. isotropic elastic materials, offer a rich variety of still unresolved problems, many everyday applications in industry and engineering—food industry, construction, mineral extraction—deal with inhomogeneous continua such as emulsions, suspensions, granular materials, etc. Spectral, in particular, vibrational, properties of these composite materials are important for quality control, suppression of vibrations, defect detection, among other practical aspects. I have been working in this area since my M.Sc. thesis [1] (supervised by Prof. A. S. Shamaev from Lomonosov Moscow State Univ—MSU, & Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences—IPMech, and advised by Prof. V. V. Vlasov from MSU). In [1], a formula for an asymptote of complex natural frequencies for a mixture of weakly-viscous fluids was derived, see Fig. 1. This study was inspired by works of V. Zhikov [SR1] on spectra of homogenized models of composite materials. Such findings may give an understanding of the oscillatory and dissipative properties of composites where the application of standard approaches. e.g. vibration theory, is hindered. Although results in [1] were obtained through simple asymptotic techniques, they led to several papers exploring the spectra of various homogenized models [2, 3, 4], and paved the way for a series of papers by A. Shamaev, V. Vlasov and their collaborators, e.g. [SR2, SR3, SR4, SR5] on rigorous analysis of spectral properties of operators arising in homogenization theory. Findings in [2, 4] explain, for example, experimental observations where a porous elastic material saturated with a viscous fluid has only a finite number of natural frequencies [SR6] contrary to the purely elastic case when it is infinite. A homogenized model of

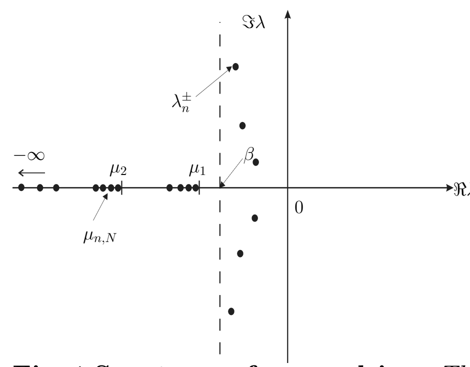

Fig. 1 Spectrum of an emulsion. *The spectrum consists of real (dissipative) series $\mu_{n,N}$ with limit points μ_N, a series going to $-\infty$, and a complex (oscillatory) series $\lambda_n^\pm$ that has an asymptote β [4].*

a mixture of two fluids employed in [1] was also studied in my Ph.D. thesis [5] (advised by A. Shamaev), where a kernel of the dynamic Darcy equation was constructed numerically and strong L_2-convergence of velocity field and its gradients was demonstrated [6] extending results of T. Levy and E. Sanchez-Palencia [SR7, SR8].

2.2 Combining experiments and vibration theory

Another approach based on experimental data was utilized in my Ph.D. thesis [5, 7, 8] in collaboration with Prof. S. V. Nesterov (IPMech & Bauman Moscow State Technical University—BMSTU) and Prof. L. D. Akulenko (IPMech & Moscow Institute of Physics and Technology—MIPT) for studying acoustic properties granular media saturated with a liquid (e.g. seafloor sand). The experimental setup for these studies, see Fig. 2, was constructed by S. Nesterov with the goal of measuring the resonant frequencies of a vessel filled with fluid and containing the sample media By analyzing these resonant frequencies, it is possible to derive explicit expressions for the dynamic density and the speed of sound in the sample by employing perturbation techniques for the analysis of the acoustic (Helmholtz) equation. The resulting formulas depend on the geometrical and physical parameters of the resonator and the frequency of the external acoustic field. Similar techniques were used in studies of the elastic media with viscous inclusions in collaboration with Prof. I. Pettersson (Narvik Univ, now Chalmers Univ of Technology & Univ of Gothenburg).

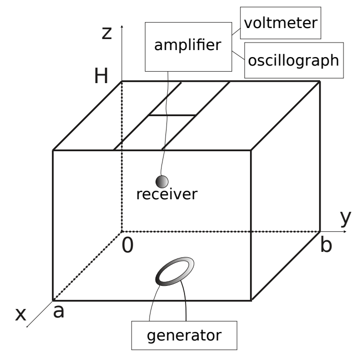

Fig. 2 Scheme of the resonator [8]. *A sample of granulated medium is placed in a vessel filled with water. The resonance frequencies are measured.*

2.3 Applications of homogenization theory: preventing sinkholes

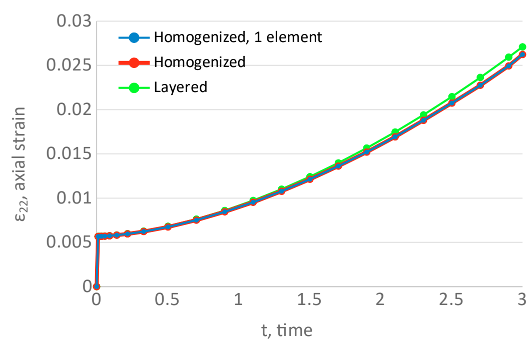

Fig. 3 Homogenized vs original models [9]. *Strains for uniaxial stretching stress applied along the layers.*

Mining, although necessary, may have serious environmental impacts. One of the repercussions of potash mining is that groundwater gradually dissolves the walls of an empty mine which suddenly collapses causing a sinkhole. To improve the sustainability of mining and reduce environmental damage, the modeling of the long-time behavior of massive soil structures is vital. In a framework of a research contract with a mining company UralChem (one of the largest producers of ammonia and nitrogen fertilizers in Russia), a computational homogenized model for layered creep materials was developed. A layered geometry was motivated by the natural sedimentary structure of potash deposits consisting of many salt layers. The model is based on Boltzmann-Volterra hereditary theory [SR9] with kernels of the Abel type. For its derivation, the homogenization approach from [SR10] was utilized. Direct solution of motion equations requires a fine mesh in each layer of salt, which essentially prevents direct simulations if many layers are involved. In contrast, a homogenized model needs a rather coarse mesh as for a uniform structure. However, the quality of the homogenized solution depends on the number of layers—the more layers, the better it approximates the solution to the original model.

A comparison of numerical results due to homogenized model with the solution to original (non-homogenized) equations of motion was performed in [9]. It was shown that even 10 layers of each material is enough to achieve an error of about 5% when the sample has been deformed by 10%-15% already, see Fig. 3. In [10], a more general model with nonlinear constitutive creep stress-strain relations was considered. For this model, a homogenization procedure as well as a numerical algorithm for finding displacements and stresses were proposed based on partial time discretization. This project was carried out by the group of A. Shamaev in collaboration with Corr. Member of RAS O. E. Melnik, Prof. V. V. Vedeneev, and members of their groups from the Institute of Mechanics at MSU and T-Platforms.

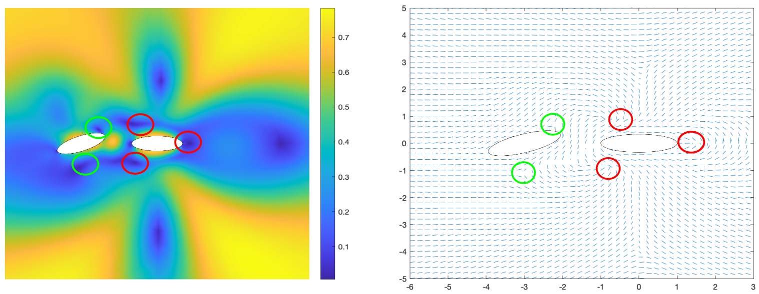

Fig. 4 Swimmers in viscoleastic liquid crystal (mucus) [11]. *A bacterium (the ellipse aligned horizontally) "catches" another one that crosses its wake. Left: scalar order parameter (anisotropy level). Right: director field (local molecular orientation). Circles mark topological defects of the liquid crystal where anisotropy is suppressed.*

2.4 Modeling bacterial motion in mucus

Recently, I have been engaged in studies of active matter, a specific type of continuum characterized by local consumption of energy and usually composed of autonomous agents. In particular, suspensions of bacteria exemplify such materials. In this research, we are interested in studying bacterial behavior while they swim in mucus, such as cervical or gastric mucus. While bacteria live throughout the human body, their main habitat is body surfaces such as skin and covered in mucus organs, e.g. gastrointestinal and reproductive tracts. To prevent bacterial infections and facilitate the proliferation of symbiotic species, we need to understand bacterial behavior. By studying bacterial motion, we may find ways to control their behavior and use this knowledge to our advantage, potentially impacting the treatment of infertility and the prevention of bacteria-born diseases by developing ways to fight pathogen invasion at mucosal surfaces.

In [11], bacteria were modeled as rigid particles while their activity was represented via so-called squirmer boundary condition [SR11, SR12]. The mucus itself was described by nonlinear Edwards-Beris PDEs for liquid crystals coupled with PDEs for conformation tensor representing viscoelastic contribution to stresses [SR13]. The computational results in [11] shed light on the onset of collective motion, such as the formation of bacterial 'trains' (as shown in Fig. 4), as well as on the peculiarities of bacterial motion in mucus, such as speeding up during back-and-forth motion. Currently, we are modifying this model to speed up computations by replacing the rigid active particles of finite size with point forces, or dipoles. This will enable us to simulate collective motion on a macroscopic scale while still tracking each agent individually. This research is performed in collaboration with Profs L. Berlyand and I. Aronson as well as members of their groups (Pennsylvania State Univ).

3 Solving Eigenproblems by Varying Length

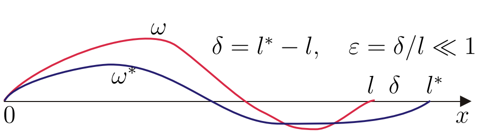

Fig. 5 Variation of the string's length. *If the length l of a string is changed slightly, then the frequency ω also changes by a small amount.*

While working on optimal control problems, it is often necessary to estimate natural frequencies of the controlled system. This is of particular importance for elongated rod-like systems such as manipulators, antennas, pipelines, and others, since their vibrational properties directly affect control efficiency, especially for precise control methods. As mentioned above, non-damaging quality control and defect detection are also related to the vibrational (acoustic) properties of a system in question. A classical way to find the natural frequencies of a vibrating object is to solve the Sturm-Liouville problem (a boundary value problem for an ordinary differential equation) or its analog.

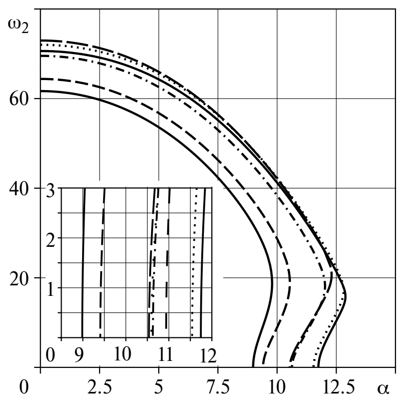

Fig. 6 Pipeline vibrations [18]. *The second natural frequency of a vibrating pipeline has two possible values for high velocities of the transported fluid.*

However, if the coefficients of the equation are non-constant (e.g., density, rigidity, or geometry vary along the length of the rod), there is no universal approach to solving such an eigenproblem. Since it is common to work with dimensionless variables, the studied system's length is usually assumed to be equal to 1, and the problem is solved on a fixed interval $(0, 1)$. However, it is a well-known experimental observation that for many vibrating systems, a small change in the length l leads to a small change in the eigenvalue λ. For example, changing the length of a guitar string causes a small change in the tone (frequency), as shown in Fig. 5. Basing on this observation, S. Nesterov and L. Akulenko had proposed [SR14] a numerical approach to solution of Sturm-Liouville problems with non-constant coefficients.

In [12, 13], we generalized the afore-mentioned method to Sturm-Liouville problems with matrix coefficients that are nonlinear in the spectral parameter (frequency). Such eigenproblems describe coupled rods undergoing longitudinal vibrations or bending vibrations of an Euler-Bernoulli beam or other similar system. The technique is based on the shooting method combined with Newton-type iterations: on each step an approximation to the eigenvalue λ_i is corrected by its derivative with respect to the interval's length: $\lambda_{i+1} = \lambda_i + \varepsilon\lambda'(l_i)$. The formula for the derivative $\lambda(l)$ is found explicitly. In [14], this approach was extended to scalar Sturm-Liouville problems with coefficients and boundary conditions depending on spectral parameter when, for instance, point masses are attached to rod's ends. The results of [13, 14] were combined in [15], and, finally, a unifying formula for $\lambda'(l)$ was derived in [16] for linear Hamiltonian systems nonlinear in the spectral parameter. The latter extension allows for finding eigenvalues and eigenfunctions of a wide class of elongated system, e.g. Timoshenko beams, pipelines, etc, provided self-adjointness and as long as the problem is self-adjoint and coefficients are reasonably smooth. Major advantages of this approach are (i) quadratic rate of convergence of involved Newton-type iterations which provides high precision of the results and (ii) a simple code implementation based on solutions of ODEs (that is, on shooting).

This approach was applied for studying natural oscillations of elongated systems in [17, 18, 19, 20, 21]. A comparison of several engineering models of bending vibrations (Timoshenko, Rayleigh and Euler-Bernoulli) was done in [17] for a circular beam with quadratically changing radius. Natural frequencies of a pipeline transporting ideal fluid based on elastic (Winkler) foundation were analyzed in [18], see Fig. 6. A rotating beam (rotor blade) with polynomial and exponential types geometries was studied in [19]. It was shown that the developed approach allows for achieving and enhancing precision previously reported in the literature with relative ease. A classical singular problem of vibrations of a Kirchhoff wedge beam was considered in [20] in the case when the beam (blade) is rotating and has a crack. Finally, in [21] the developed method was combined with experimental techniques which allowed for identification of defects (cracks) of a vibrating rod.

4 Control Problems for Mechanical Systems

4.1 Equipment for crystal growth at the International Space Station

One of the difficulties encountered during crystal growing is the variation in the natural gravitational field. That causes undesirable inclinations in the crystal axis. This issue can be resolved by moving manufacture into space where gravity is very small. Theoretically, this allows to grow larger pure crystals needed, for example, in semiconductor industry. However, the non-uniform micro-gravitational field is still present on orbital stations due to residual gravity as well as due to motion and operation of the station.

The group developed control algorithms for a two-axis gimbal for a vibroprotective platform installed at the Russian section of the International Space Station (ISS) to reduce undesirable vibrations.

Our group developed control algorithms for a two-axis gimbal for a vibroprotective platform, see Fig. 7, supposed to be installed at the Russian section of the International Space Station (ISS). In [22], a kinematic control was considered. Quasi-optimal control feedback control laws were proposed in [23] for a system with one degree of freedom. In [24], the control problem for the rotatory platform with two degrees of freedom is reduced to the control problem for an ODE system with nonlinear friction in the presence of uncertainties and an appropriate feedback control as well a filtration algorithm of the incoming data were proposed. As far as I know, the vibroprotective platform is currently operational at the ISS. This project was carried out in collaboration with Dr. A. E. Borisov (Central Research Institute of Machine Building—TsNIIMash), L. Akulenko and Corr. Member of RAS N. N. Bolotnik (IPMech, MIPT). For the paper [22], the authors were awarded with Nauka-Interperiodika (the publisher of Russian Academy of Sciences' journals) prize.

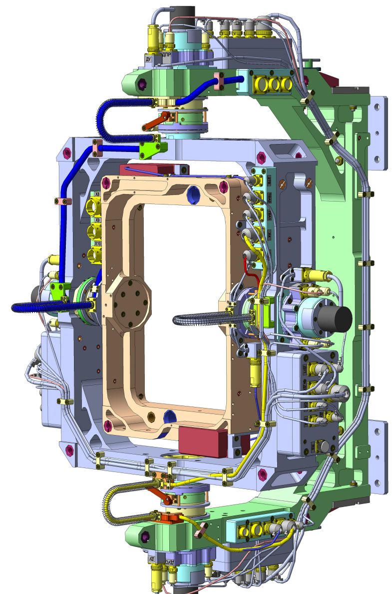

Fig. 7 Two-axis gimbal [24].

4.2 Controlling temperature distribution via thermoelectric converters

Fig. 8 Experimental setup: a thin Peltier element b/w two aluminum cylinders. *The cylinders are thermally insulated at the ends, the temperature is measured at several points on cylinders' sides.*

With the development of microelectronics, some well-known physical phenomena have found their way into everyday practice. The thermoelectric effect, which encompasses the Peltier, Seebeck, and Thompson effects, has been widely used in engineering through the implementation of thermocouples, thermoelectric generators, and coolers. These devices are responsible for converting heat into electricity and vice versa. In short, the Peltier effect produces heating or cooling at an electrified junction of two conductors with different physical parameters (known as Peltier coefficients). The Seebeck effect works in the opposite direction, causing electric current in the presence of a temperature difference, while the Thompson effect creates an additional heat flux if the Seebeck coefficient depends on temperature. Various devices based on these effects (e.g. Peltier elements, also called Peltier devices, coolers, etc.) are used in laboratory equipment, industry, and consumer products as miniature coolers and temperature controllers [SR15, SR16]. Examples of these devices include those used in the Mars Curiosity rover, telescopes, dehumidifiers, water collectors, and beverage coolers.

A Peltier element (PE) is typically a thin, flat device consisting of small "thermoelectric legs" connected in series and placed between two plates. When an electric current is applied, one plate is heated while the other is cooled, and if the current direction is reversed, the roles of the plates swap. Additionally, if one side is heated or cooled by an external source, an electric current will appear in the circuit, which leads to coupling between the controlled system and the actuator, with electricity producing heat and vice versa. Initially, Prof. G. V. Kostin (IPMech, MIPT) and I considered a simplified linear model without coupling. In [25], we used the heat flux as a control input to develop an explicit LQR-type feedback control of the lowest eigenmodes for a cylinder actuated from the top and bottom sides. This eigenmode approximation was enhanced in [26], where we utilized polynomial approximations of the temperature and proposed an LQR feedback control strategy that minimizes both the deviation from the desired distribution and the approximation error.

Next, in collaboration with Prof. Dr.-Ing. H. Aschemann (Univ Rostock—UR) and Prof. Dr.-Ing. habil. A. Rauh (UR, now Carl von Ossietzky Univ of Oldenburg), we developed a control-oriented model of a PE [27, 28]. The PE is placed between two solid bodies, one treated as a heat sink and the other as a working body in which a certain temperature distribution is desired. The heat processes in these bodies are described by standard heat equations, while the heat equation for the PE contains a nonlinear term depending on both temperature and control voltage. The three equations are coupled by boundary conditions on the sides of the PE that include another nonlinear term.

For an experimental cylindrical setup shown in Fig. 8, identification of parameters and model validation were performed in [29] by using FEM and in [30] by means of model order reduction. In [27], stationary states of the considered setup were investigated. A partial feedback linearization that makes PDEs linear in temperature (although still nonlinear in the control voltage) was proposed in [31]. Also in [31], a combined control strategy that exploits the feedback linearization and piecewise constant feedforward control of the system's lowest eigenmodes was proposed. The optimal signal was computed via gradient descent. This strategy was enhanced in [32] by compensating for the varying ambient temperature (of the environment) via an additional feedback term in the control signal. Constrained control was studied in [33], where a penalty cost functional was proposed to enforce the constraints. Finally, the minimal control time was estimated in [34].

4.3 Motion control for elastic systems and smart structures

Elongated elastic systems, such as manipulators, machine parts, and construction elements, have long been one of the foci of applied control theory due to their importance in practical applications. In [35], the method of solving eigenproblems described in Sect. 3 was used to propose an LQR-type control approach for a vibrating non-uniform string (a chain) with a load at the end, such as one hanging from a crane. This strategy was improved in [36, 37] by incorporating a "slow" feedforward control input that asymptotically does not excite vibrations. In [38], a combination of eigenproblem solution and the "slow" control was proposed for a beam with an attached tip mass (a manipulator with a load) moving in a plane and undergoing bending vibrations.

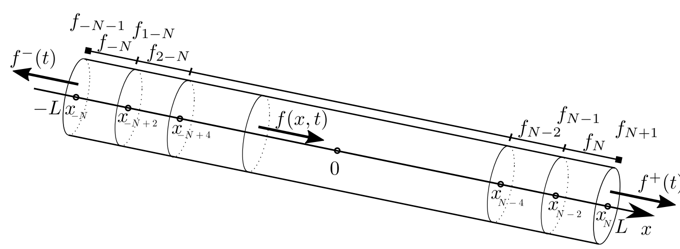

Fig. 9 Scheme of a rod controlled by boundary ($f_{\pm N\pm 1}(t)$) and distributed ($f_{-N+1}(t), \ldots, f_{N-1}(t)$) inputs. *The piezoelectric force $f(t,x)$ acts effectively through the force jumps $f_{-N}(t), f_{-N+2}(t) \ldots, f_N(t)$*

Recently, in collaboration with G. V. Kostin, we began studying elongated elastic systems controlled by finite-dimensional distributed in space inputs. While lumped boundary inputs are rather suitable for practical implementation using drivers (e.g. attached at the ends of a rod), the distributed inputs provide a substantial mathematical advantage. However, it is more challenging to apply them in practice because they require continuous force along the rod. Thus, we assume from the beginning that the distributed force is piecewise constant (i.e. finite-dimensional) in space, having in mind implementation via piezoelectric actuators. That is to say, identical piezoactuators are placed along the rod symmetrically (e.g., in pairs) such that there is no empty space between them. These setups are sometimes referred to as "smart structures" [SR17]. For simplicity, we do not consider a particular model of piezoelements (e.g., the IEEE standard model [SR18]) understanding the distributed control inputs as additional normal forces acting on the rod's cross-section.

In a preliminary study [39], we developed a feedforward control approach for a rod undergoing longitudinal vibrations without piezoelements actuated by boundary inputs only. By utilizing the method of integro-differential relations [SR19], the wave equation in [39] was replaced with a variational problem in the time-space domain. The latter problem serves as a constraint in the optimal control problem of minimizing the mechanical energy stored in the rod during motion while reaching a prescribed terminal state. A peculiarity of our approach is that we use two unknown variables, namely, displacements v and so-called dynamic potential r, which are related via differentiation with momentum density and normal force in the cross-section. Such an approach allows us to relax the smoothness conditions on the initial data while still finding an optimal trajectory among continuous functions. The optimal control is found explicitly by solving the Euler-Lagrange equations. To this end, we introduce a mesh induced by characteristics on the time-space domain. The unknown functions v and r are represented as linear combinations of traveling waves. The boundary conditions interweave these traveling waves with the control inputs. The minimization problem for the mechanical energy is rewritten in terms of the traveling waves and results in the Euler-Lagrange equations. Finally, the minimal control time is determined. In [40] this control strategy was enhanced by additionally minimizing the energy norm of control forces.

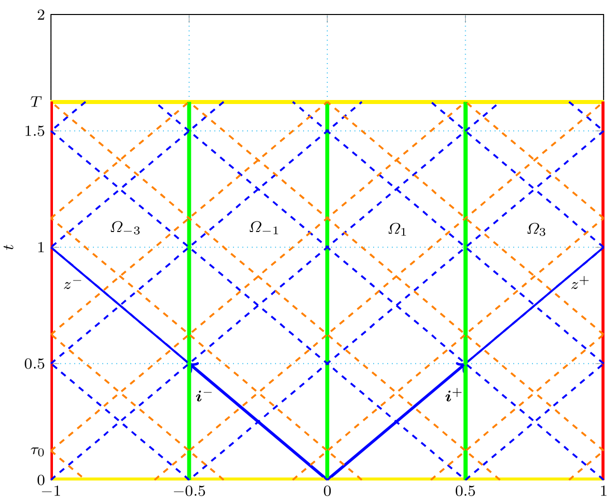

Fig. 10 Mesh on the time-space domain. *Additional characteristics emerge due to actuators [41].*

In [41, 42, 43, 44] this technique has been further enhanced to account for distributed inputs (i.e., piezoelements). A detailed description can be found in [41]. The mesh on the time-space domain now incorporates additional characteristics that emerge due to distributed actuators, see Fig. 10. Correspondingly, a larger number of traveling waves are defined, and new interelement continuity conditions arise that involve distributed inputs (more precisely, their jumps, see Fig. 9). The interelement, initial, terminal, and boundary conditions lead to an overdetermined linear algebraic system. This system is explicitly resolved for terminal times greater than a minimal one in terms of control functions and some of the traveling waves. The solution is plugged into the objective functional, and then the Euler-Lagrange equations are integrated also explicitly. A particular case where the control time is a multiple of the piezoelement's length is considered in [42]. Another special case is studied in [43], where the rod with free ends (actuated by only the distributed inputs) is studied, and the terminal state is assumed to be periodic. In [44], the squared norm of the control functions is minimized in addition to the mechanical energy.

Since piezoelements may serve as both actuators and sensors, a feedback control strategy was studied in [45], where the eigenmode approach was utilized. It was shown that the original continuous system splits into a finite number of independent groups of modes, and each of these groups is controlled by a specific linear combination of inputs. An LQR-type feedback control algorithm was proposed that minimizes the amplitudes of the lowest eigenmodes in each group (except for the motion as a rigid body). The optimal control functions are found explicitly. To estimate from above how the highest modes are actuated by the optimal signal, an asymptotic feedback control was proposed.

5 Application of The Diffraction Theory: Manufacturing Semiconductor Devices

In the project devoted to the development of alternative methods for producing integrated circuits (ICs) in the microelectronics industry, a practical application of the diffraction theory was undertaken in the framework of a startup founded by Prof. V. Rakhovsky (Nanotech SWHL GmbH). One of the main steps in IC production involves projection of the desired IC pattern through a mask onto a wafer covered with light-sensitive material (photoresist) [SR20]. The idea we implemented was to replace the usual projective mask with a holographic mask. The projective mask, in the simplest case, repeats the desired pattern. In contrast, the holographic one is a result of interference of the light from the pattern and a coherent light source and does not have clear similarities with the original image [46]. The holographic mask creates 3D patterns in a single exposure and is cheaper in production being highly sustainable and low sensitive to defects (see Fig. 11) which are one of the major issues in mask exploitation. Due to diffraction, an aerial image that a mask produces does not exactly repeat the desired pattern. In projective lithography, this problem is resolved by correcting the mask physically (adding small elements and complex supplementary layers).

The Sub-Wavelength Holographic Lithography (SWHL) addresses this issue computationally [47]. The holographic mask is optimized during its computation. As a result, the manufacturing process of the optimized mask is not different from the production of the non-optimized one [48]. Both are represented via a number of holes in the non-transparent layer [49]. Even if supplementary layers are needed, they are much simpler than in projection lithography. It is also easier to position the holographic mask in the optical scheme [50]. During my involvement in this project, we developed appropriate numerical software for calculating holograms based on modified Fourier transform for experimental validation and for the optimization of holograms by utilizing gradient descent and local variation methods. The results of experimental validation of this software were published in [SR21, SR22]. In addition to [49, 50], the invention was also patented in [51, 52, 53, 54]. Nowadays, the startup company Nanotech SWHL GmbH continues its work in Zurich, Switzerland, in collaboration with Swiss Federal Laboratories for Materials Science and Technology (Empa), Swiss National Supercomputing Centre (CSCS), Fraunhofer Institute for Electronic Nano Systems, JEOL Ltd, and CEA-Leti.

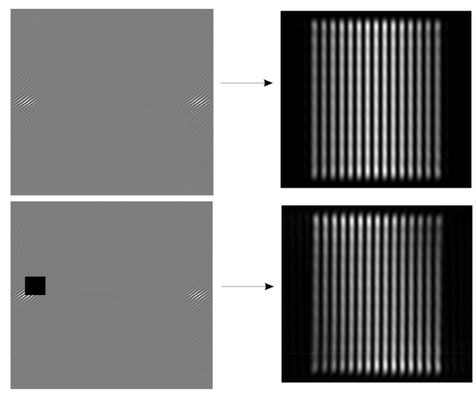

Fig. 11 Holographic mask and an image it produces. *A relatively large defect—black square—on a holographic mask (left) almost does not affect the resulting image (right) [48].*

6 Future work

- In light of the overview of **mechanics of inhomogeneous continua** provided in Sect. 2, the study of bacterial motion in mucus presents a wide range of research opportunities. Currently, I am developing a model and respective computational algorithms to capture both individual and collective motion. I plan to utilize this approach to (i) model "ant trails" when several bacteria follow a leading one that burrows a tunnel in mucus; (ii) simulate memory effects such as "stop-and-go" motion when a bacterium slows downs and then speeds up if stresses overcome certain threshold; and (iii) identify spatial and temporal scales of collective motion. I also expect that these algorithms can be used for numerical homogenization of bacterial suspensions to reveal, for example, how bacteria affect the elastic properties of mucus.

- I plan to extend the **eigenproblem solution** algorithm described in Sect. 3 to 2D vibrating elastic systems such as rectangular plates. This can be accomplished by employing an "iterative" separation of variables approach: virtually cutting the plate into strips, solving a number of eigenproblems first in one direction and then in another direction until the resulting approximation is close to the exact solution. Another direction of studies is to overcome the numerical difficulties of dealing with high-frequency oscillations for beams and rods, where the shooting requires too high precision.
- Regarding **control problems in heat transfer**, see Sect. 4.2, I plan to continue this research in two directions: (i) controlling 2D and 3D temperature distribution in solids by means of several Peltier devices attached, (ii) studying a solid "heat pump" that does not require a liquid refrigerant and consists of several cylinders connected in series with Peltier devices between them. While direction (i) may be more interesting for specific applications, direction (ii) is more promising in terms of analysis. Such a system is essentially one-dimensional, which makes it possible to study the control problem in terms of eigenfunctions. Furthermore, homogenization techniques can be applied if the number of cylinders and Peltier actuators is large enough.
- Finally, I plan to use approaches developed in Sect. 4.3 to (i) combine feedforward [41] and feedback [45] controls for the longitudinal motion of a rod; (ii) develop feedforward control strategies for bending vibrations of beams and motion of inhomogeneous rods; and (iii) study the limiting behavior of vibrating elastic systems controlled by distributed inputs. Direction (i) is conditioned by the necessity to compensate for disturbances during the optimal motion. While an analytical solution based on traveling waves developed in [41] cannot be applied directly for (ii), a similar approach based on a finite element mesh on the time-space domain may still be used. The direction (iii) is particularly interesting since it may answer questions on how well a theoretical infinite-dimensional distributed control input can be approximated by a finite-dimensional one and what functional properties the limit of the finite-dimensional input has when the number of control elements goes to infinity. In the latter case, homogenization techniques may also be applied.

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