AERSP 304 Final Exam Review

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Spring 2014

We’re here to help!

- Dr. Melton
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We’re here to help you!

The Final Exam

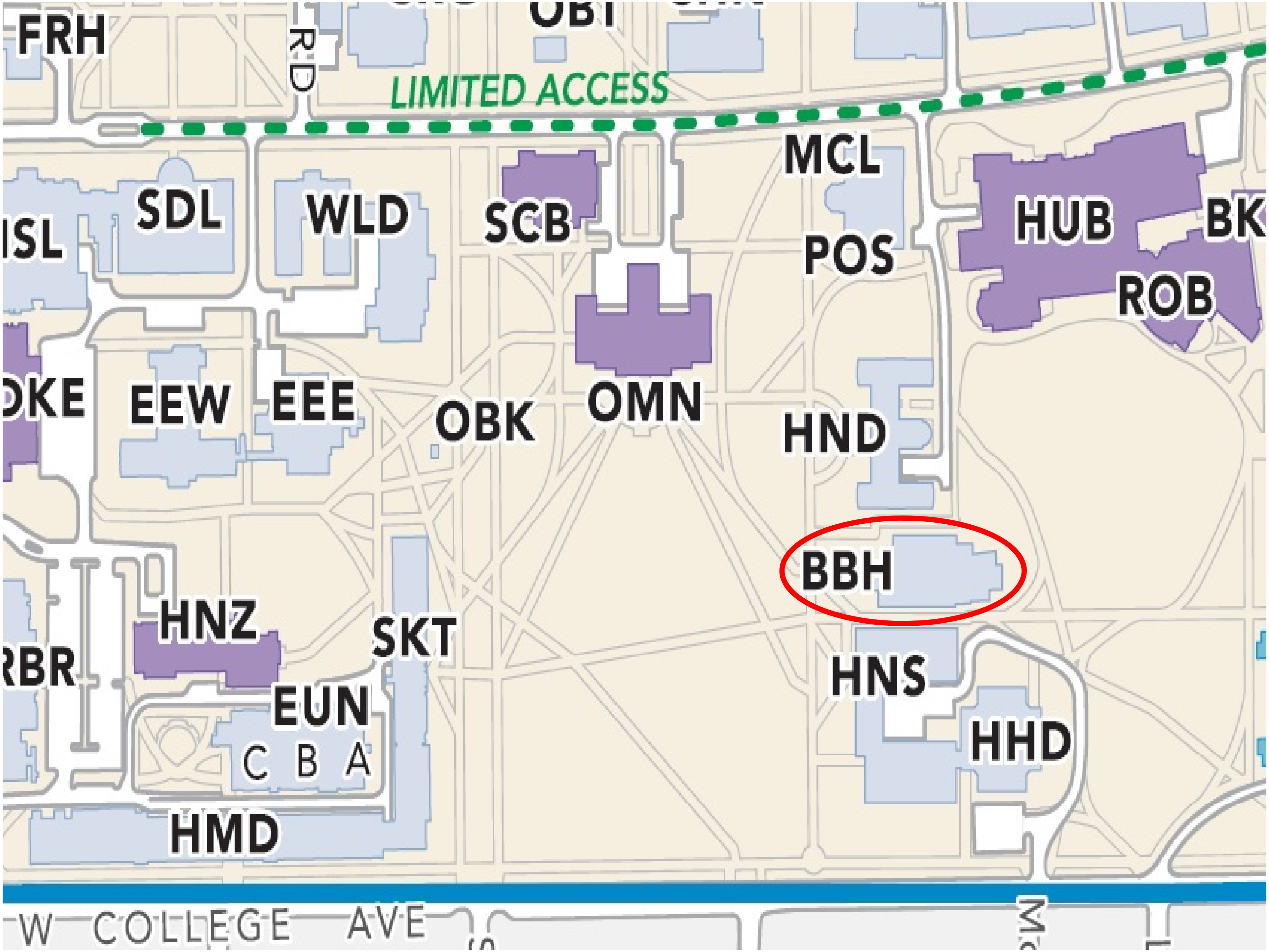
- Look over everything you know and have ever done in this class – quizzes, sample problems, readings, etc.
  - The syllabus is a good place to start if you're trying to remember everything we've covered.
  - Just because I did or did not put something in this PowerPoint doesn't mean it is or is not fair game. Try as I may, I can't read Dr. Melton's mind!

- Make use of our office hours!

The Final Exam (Cont'd)

- Reminders:
  - We drop your lowest quiz score
  - We do not curve either the exams or the course
- Ask questions if you need help! It's never too late* to ask a question about course material!
- Time and Location: Wednesday, May 7th, 10:10 – Noon in 22 BBH Building
- Exam covers everything in the course!

*Until we hand you the final exam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What is Dynamics?

- Dynamics: The exchange, dissipation and addition of total energy
- Can be stored in different forms:
  - Kinetic: Mass, Rotational
  - Potential: Springs, Structures, Gravity Fields
- Energy can also be dissipated:
  - Friction, drag, damping, etc.

- Equations of Motion (EOMs): The differential equations that describe the motion of a body or a dynamical system.

- Response: The solution to the differential equations (EOMs). The number of EOMs is equal to the number of DOFs for any system!

A note about linearity

- Most (interesting) systems are non-linear
  - Why? Because the real world is complicated…
  - Non-linear problems are generally very difficult to solve analytically and the principle of superposition generally does not hold for non-linear systems.
- How do we over come this?
  - Special Cases
  - Linearize the system
    - Binomial Expansions
    - Series Expansion (e.g. Power Series, Taylor Series, etc.)
  - Solve it numerically (solvers and methods include ODE45, Gauss-Jackson, Runge-Kutta, etc.)

Unit Impulse

- By definition

$$F(t;a) = \begin{cases} 0, & t \le \frac{-a}{2} \\ \frac{1}{a}, & \frac{-a}{2} < t < \frac{a}{2} \\ 0, & t \ge \frac{a}{2} \end{cases}$$

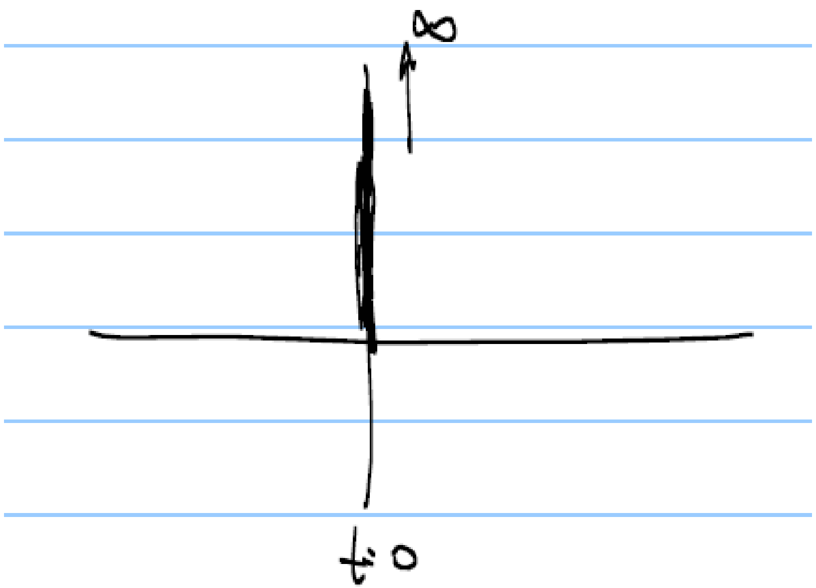
- As a gets smaller, it approaches the Dirac Delta function

$$\lim_{a \to 0} F(t;a) = \int \delta(t)\, dt$$

Dirac Delta

- Dirac Delta's are impulse functions

$$\delta(t) = \begin{cases} 0, & if\ t \neq 0 \\ \infty, & if\ t = 0 \end{cases}$$



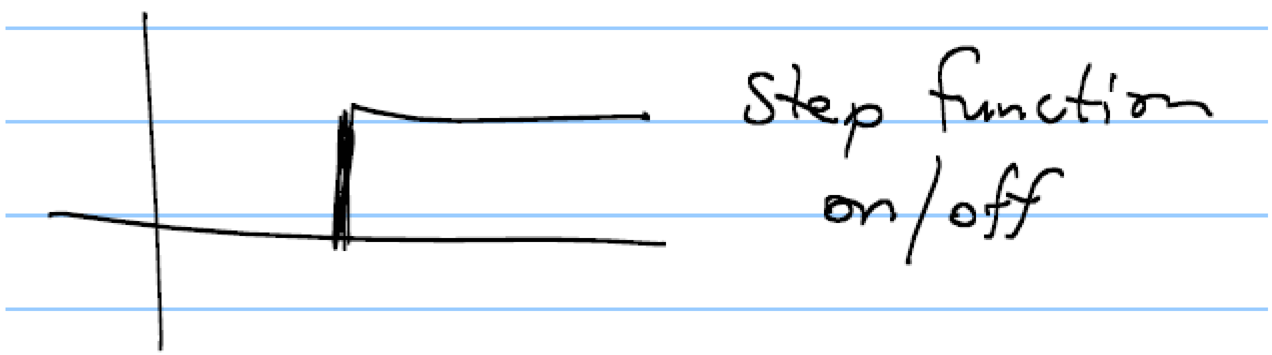
Step function

- Definition

$$a\,u(t - t_o) = a \int_{-\infty}^{t} \delta(t - t_o)\,dt$$

$$a\,u(t - t_o) = \begin{cases} 0, & if\ t < t_o \\ a, & if\ t > t_o \end{cases}$$

- Like in the homework, we can use step functions to create ramp functions



Some old AERSP 309 concepts

- Particle Dynamics

$$\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$
$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}$$
$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$$

- Linear Momentum

$$\vec{L} = m\vec{v}$$

- Angular Momentum

$$\vec{H} = I\vec{\omega}$$

- Deformable Body

$$\dot{\vec{r}}_{B/A} \neq 0$$

Vector Derivatives

- This slide was copied from my AERSP 309 final exam review (this is still important material, though!)

- Derivatives of vectors in other frames of reference
  - First Derivative

$$\frac{{}^{I}d\vec{r}}{dt} = \frac{{}^{B}d\vec{r}}{dt} + \vec{\omega}^{B/I} \times \vec{r}$$

  - Second Derivative

$$\frac{{}^{I}d^2\vec{r}}{dt^2} = \frac{{}^{B}d^2\vec{r}}{dt^2} + \frac{d\vec{\omega}^{B/I}}{dt} \times \vec{r} + 2\vec{\omega}^{B/I} \times \frac{{}^{B}d\vec{r}}{dt} + \vec{\omega}^{B/I} \times (\vec{\omega}^{B/I} \times \vec{r})$$

Strategy: Never compute anything twice!

Some E MCH Concepts

- Center of Mass (not to be confused with the center of gravity)

$$\bar{r} = \frac{1}{m}\int_{m} \vec{r}\, dm$$

- Moments of Inertia – Can be tabulated into Inertia Tensors (Matrices)

$$I_{AA} = \int_{m} r^2\, dm$$

- Parallel Axis Theorem

$$I_{total} = \sum_{i=1}^{N} (I_{i,cm} + m_i r_i^2)$$

Degrees of Freedom (DOF)

- Particle

$$n_p = 3 - k$$

- Rigid Body

$$n_p = 6 - k$$

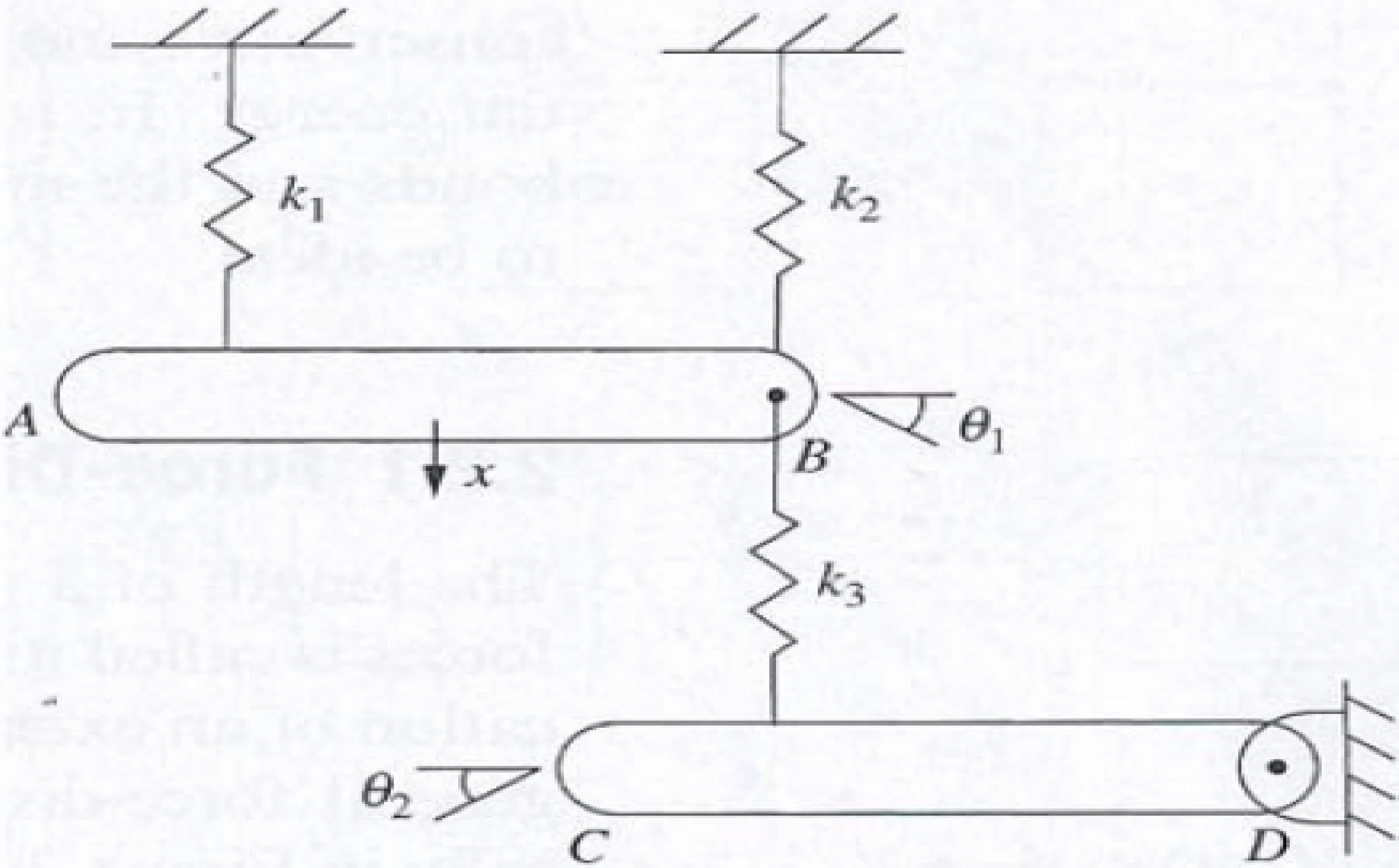
- Total DOF for a System

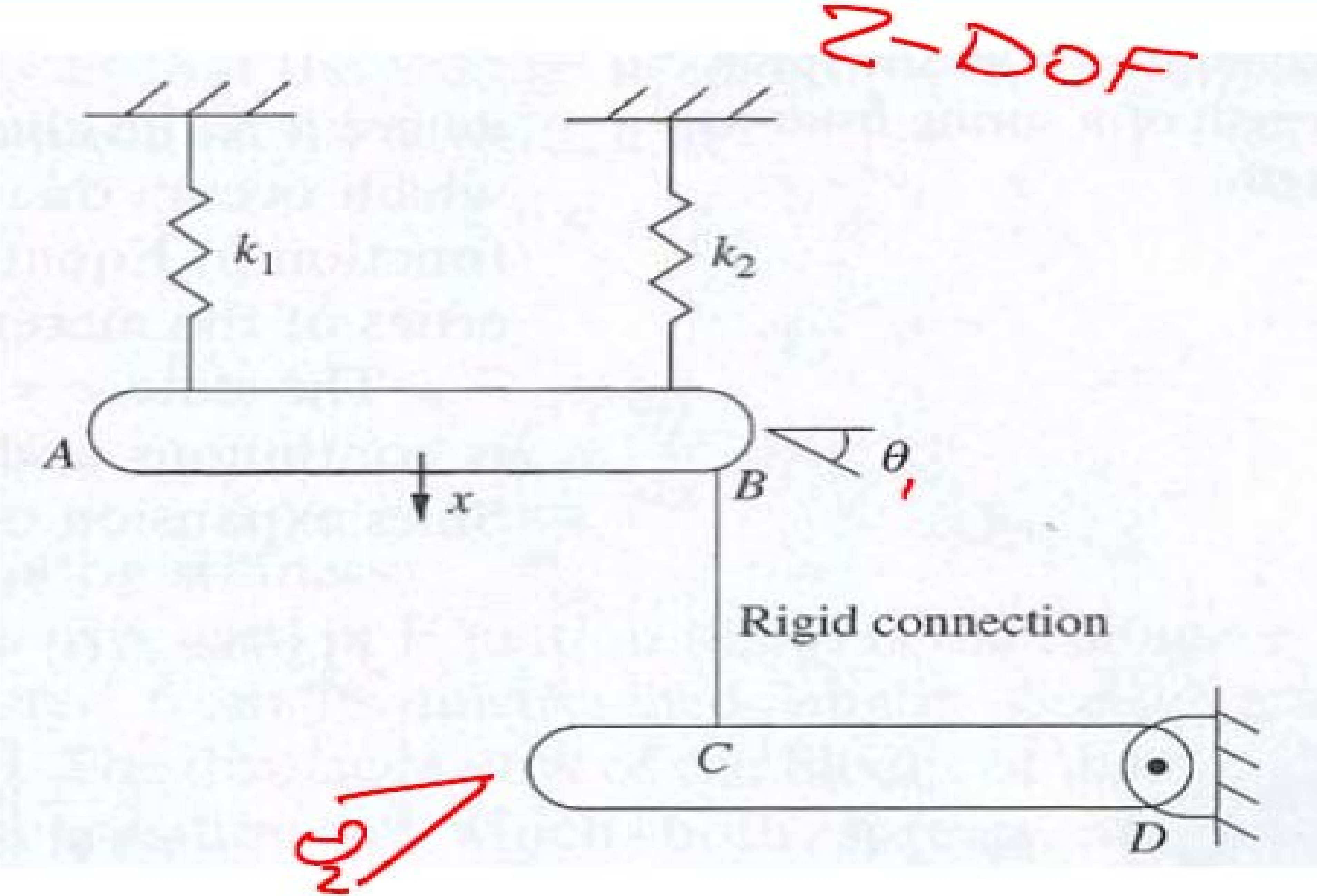
$$n_{p,Total} = \sum_{i}^{N} n_{p,i}$$

k is the number of constraints

Examples

3-DOF





2-DOF
k_1
k_2
A
B
θ_1
x
Rigid connection
C
D

Relationships

- Some useful relationships include:

$$F(t) = k\Delta x$$

$$F(t) = c\Delta v$$

$$F(t) = I\,\delta(t - t_o)$$

$$F(t) = F_o\,u(t - t_o)$$

$$M_A = -K_t\theta$$

Combinations

- Springs

Parallel: $k_{eq} = \sum_{i=1}^{N} k_i$

Series: $k_{eq} = \frac{1}{\sum_{i=1}^{N} \frac{1}{k_i}}$

- Dampers

Parallel: $c_{eq} = \sum_{i=1}^{N} c_i$

Series: $c_{eq} = \frac{1}{\sum_{i=1}^{N} \frac{1}{c_i}}$

- Beware of springs or dampers that look to be in series but act in parallel (and vice-versa)

D'Alembert's Principle

- D'Alembert's Principle allows you to convert a dynamics problem into a statics problem. Statics problems are usually easier to solve.

$$\sum \vec{F} = m\vec{a} \;\rightarrow\; \sum \vec{F} - m\,{}^{I}\vec{a} = 0$$

$$\sum \vec{M} = \frac{d\vec{H}}{dt} \;\rightarrow\; \sum \vec{M} - \frac{{}^{I}d\vec{H}}{dt} = 0$$

Free Body Diagrams

- These should be familiar from high school physics, PHYS 211, E MCH 210 (or E MCH 211), E MCH 212, etc. so I won't belabor the point too hard.
- Draw blocks to represent your figures
- You need some sort of sign convention
- Draw the directions in which the forces and/or moments are acting

$$\sum F: m\ddot{x} = F_1 + F_2 + \ \dots$$

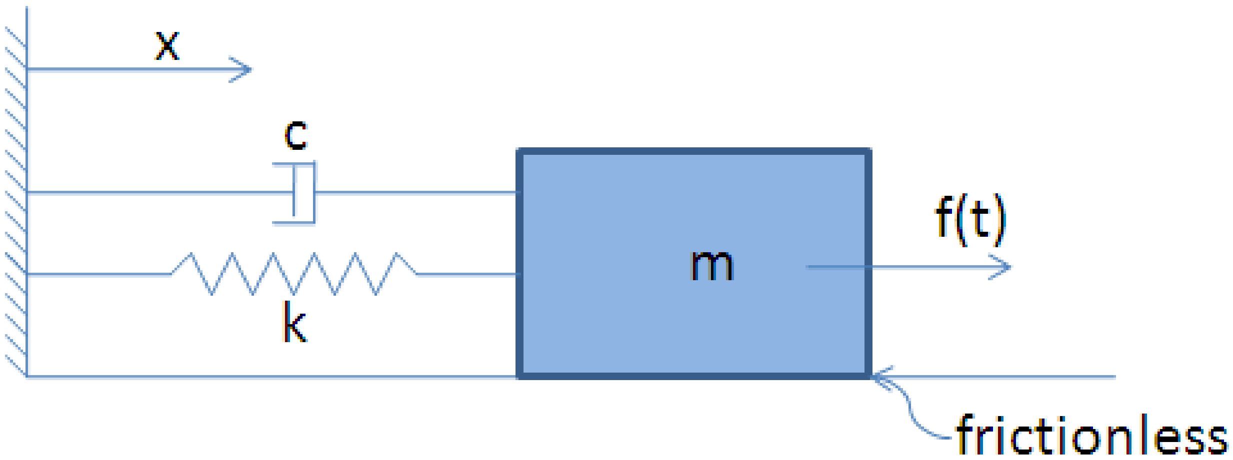
$$\sum M: I\ddot{\theta} = M_1 + M_2 + \ \dots$$

Remember: Moments and forces are related by the equation $M = Fd$ (where d is your moment arm)

Spring-Mass-Damper

- One of the most basic systems is the spring-mass-damper system

$$\sum F_x:\ m\ddot{x} = -kx - c\dot{x} + f(t)$$



Always watch your sign conventions!

Single vs. Multiple DOF Systems

- SDOF System

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

- MDOF System

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + Kx = \vec{F}(t)$$

Be able to put MDOF Systems into matrix form! This shows up throughout the course (e.g. forming impedance matrices, etc.)

Lagrange's EOMs

- Suppose we have an *n*-DOF system, then

T = Kinetic Energy of the System

V = Potential Energy of the System

Q_i = generalized external force

We can then form the Lagrangian

$$\mathrm{L} = T - V$$

Lagrange's EOMs Cont'd

For *i = 1, 2, … , n*

Non-Conservative Systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = Q_i$$

Conservative Systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0$$

What is Q_i?

- Q_i contains the generalized forces or moments that were not included in the Lagrangian.
- Friction, forcing function, and any other velocity-dependent terms (including damping and drag) are included.
  - In other words, non-conservative effects are included
- We calculate Q_i using the principle of virtual work.

Principle of Virtual Work

- Imagine the system is in motion ($q_i, \dot{q}_i$ are nonzero for *i* = 1,…, *n*).
- At some time *t*, displace q_i by δq_i
- Then the virtual work done by the Q_i is

$$\delta W = \sum_{i=1}^{n} Q_i \delta q_i$$

and

$$Q_i = \frac{\delta W}{\delta q_i}$$

- Free body diagrams can help you figure out what is happening with the generalized forces.

Laplace Transforms

- These apply only to linear systems (i.e. the EOMs must be linear)
- Laplace transforms permit us to solve (relatively difficult to solve) differential equations by converting them into (relatively simple to solve) algebraic equations.
- This gives us a more mechanical (procedural) approach to these problems (as opposed to the more special case ("here's how I approach this one particular kind of problem") approach that many of you might have struggled with in MATH 250 or MATH 251)

Laplace Transforms Cont'd

- Laplace Transforms can often be combined with other techniques such as partial fraction expansions or partial fraction decompositions
- Definitions:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds = f(t)$$

- In practice, we don't use these definitions (some of you tried to brute force these on the first midterm) – use look-up tables instead! Sometimes you have to manipulate the expression to make it match up with what is in the table – this is generally not too bad to do.

The Beautiful Properties of the Laplace Transformation

- Linearity (principle of superposition)

$$\mathcal{L}\{a\, f_1(t) + b\, f_2(t)\} = a\, F_1(s) + b\, F_2(s)$$

- Differentiation

$$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = s\, F(s) - f(0)$$

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = s^2 F(s) - s\, f(0) - f'(0)$$

- Integration

$$\mathcal{L}\left\{\int f(t) dt\right\} = \frac{F(s)}{s}$$

- Final Value Theorem

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s\, F(s)$$

Partial Fraction Decomposition

$$X(s) = R(s)\frac{N(s)}{D(s)}$$

$R(s)$ is a non-polynomial function of s

$N(s)$ is a numerator polynomial of order p

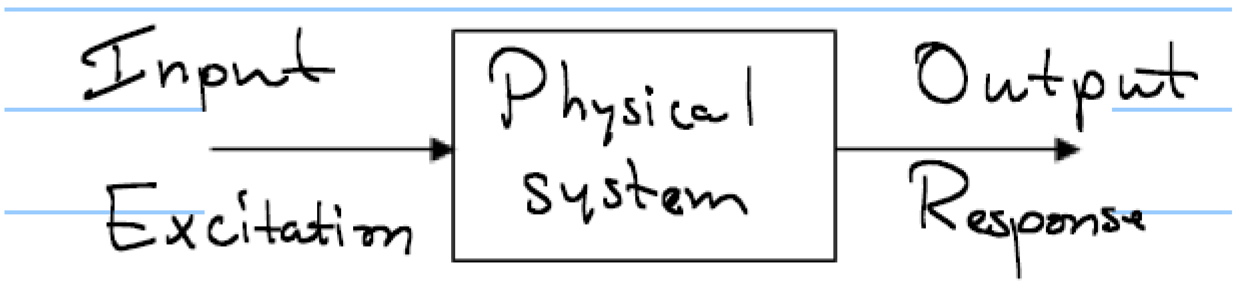
$D(s)$ is a denominator polynomial of order n

For a physical system, $n \geq p$ but the case of $p > n$ rarely happens. (If it does, Google™ it.)

Partial Fractions Cont'd

- There are several cases you should be aware of (and know how to deal with)
  - Distinct, real roots
  - Complex roots (will always appear in complex conjugate pairs)
    - Approach 1: Don't factor it (solve for two unknowns)
    - Approach 2: Treat as distinct roots
    - Try using phasors! The math is much easier in many cases than trying to brute force the algebra.
  - Repeated real roots
  - Combinations of these (you saw something like this on the second midterm)

Transfer Functions



$$G(s) = \frac{\mathcal{L}\{x(t)\}}{\mathcal{L}\{f(t)\}} = \frac{X(s)}{F(s)}$$

More generally, we can form a matrix

$$\vec{X}(s) = \underline{K}(s)\vec{F}(s)$$

Remember what we did to find the system response if we had the transfer function and knew what the input was.

A Couple of Strategies for Working with Transfer Functions

- Polar form of complex numbers – sometimes is easier to work with mathematically
- Partial fraction decomposition – makes nasty fractions into smaller, easier to manage ones
- Laplace table – if your *s*-domain result looks like something from the table, use the table!
- Take advantage of mathematical tricks such as complex conjugates to save precious time
- Watch any/all quadrant checks!

System Inputs

- There are several different commonly seen inputs to a system
  - Free Response: $F(t) = 0$
    - Typically with non-zero initial conditions.
    - Sometimes we'll say "ignore initial conditions" – in this case, assume all necessary initial conditions are zero. You need to understand what's going on with the transfer functions!
  - Step input: $F(t) = Au(t - t_o)$
  - Impulse input: $F(t) = B\delta(t - t_o)$
  - Ramp Response: $F(t) = C\left(t - t_{0,1}\right)u(t - t_{0,2})$

Transient Response

- Some key definitions to know – these are all important characteristics of a system's response
  - Final Value: The steady state or final value of the response of the system (can often be found via the final value theorem)
  - 2% Settling Time: Time it takes for the response to enter and stay within 2% of the final value
  - 10-90% Rise Time: Time it takes for the response to go from 10% to 90% of the final value
  - Percent Overshoot: Percent of the max load of the input, calculated from the equation

$$\eta_o = 100\% \frac{x_{peak} - G(0)}{G(o)}$$

  - Know the approximation formulas from the notes

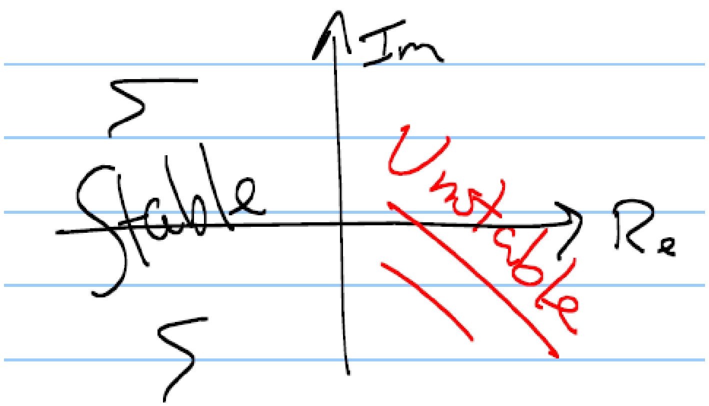
Stability Analysis

- Couple different techniques
  - Look at the poles of the system
  - Routh's Table/Criterion
    - Number of sign changes is equal to the number of poles with positive real parts. We made heavy use of these!
    - Graphical methods such as Root Locus, Bode plots or Nyquist plots. We talked about the first two of these, but be aware there are others used in the real world.

- Stability Classifications
  - Asymptotic stability
    - Stable (decays to zero)
    - Neutrally Stable (steady state) – often rather sensitive to perturbations that can lead to instability
    - Unstable (blows up!)
  - Stability isL (in the sense of Lyapunov)
  - Many more definitions of stability exist in the literature.

Stability and Relative Stability

- To make a system stable, you want all of the poles to be in the left half plane (*i.e.* have negative real parts)
- We can also say a system is stable relative to some (arbitrary) condition



Root Locus

- Transfer function of the form

$$G(s) = \frac{X(s)}{F(s)} = \frac{N(s)}{D(s)}$$

- The denominator can be rewritten as

$$D(s) = Q(s) + KR(s) = 0$$

- So what's going on?
  - $K = 0 \rightarrow$ Poles are roots of Q
  - $K = \infty \rightarrow$ Poles are roots of R
  - Look at the Root Locus plot – see where the plot crosses the imaginary axis; look at the gain value K

Second Order Systems

- Standard form of a differential equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}f(t)$$

- Transfer Function

$$G(s) = \frac{As + B}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Motion Input

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 2\zeta\omega_n\dot{y} + \omega_n^2 y$$

- Poles of the transfer function

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Second Order Systems Cont'd

- Undamped free response

$$x(t) = Asin(\omega_n t + \phi)$$

- Underdamped free response

$$x(t) = A_d e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

- Critically damped free response

$$x(t) = e^{-\omega_{nt}} + (\dot{x}_o + \omega_n x_o) + e^{-\omega_{nt}}$$

- Overdamped free response

$$x(t) = \frac{1}{2\sqrt{\zeta^2 - 1}}\left\{\left[x_o\left(-\zeta + \sqrt{\zeta^2 - 1}\right) - \frac{\dot{x}_o}{\omega_n}\right]e^{s_1 t} + \left[x_o\left(\zeta + \sqrt{\zeta^2 - 1}\right) + \frac{\dot{x}_o}{\omega_n}\right]e^{s_2 t}\right\}$$

Higher Order Systems

- Recall that we write these equations as

$$M\ddot{\vec{x}} + C\dot{\vec{x}} + Kx = \vec{F}(t)$$

- We can then form the impedance matrix

$$Z(s) = Ms^2 + Cs + K \rightarrow Z(s)X(s) = F(s)$$

- Frequencies are found from the roots of the determinant equation

Systems with Time Delay

- Response at a time t is affected by the system's response at a previous time $t-\tau$ for a fixed value of τ
- Use a modified version of the second shifting theorem

$$\mathcal{L}\{x(t-\tau)\} = e^{-\tau s}x(s)$$

State Space and Numerical Integration

- It's hard to test you on numerical integration in a class like this. Instead, Focus on knowing how to put equations in State Space form
  - We use State Space form for numerical integration and control analysis
  - In essence, you convert higher-order linear or non-linear differential equations into first order differential equations (which in theory are easier to solve)
  - No derivatives on the right hand side!

State Space Example

- Assume we have an equation of the form

$$A\dddot{y} + B\ddot{y} + C\dot{y} + Dy = f(t)$$

where *A*, *B*, *C* and *D* are constants $\epsilon\ \mathbb{R}$

Dependent variable: y
Highest derivative: 3

1 state *x* 3 derivatives = 3 equations

State Space

Let

$$x_1 = y$$
$$x_2 = \dot{x}_1 = \dot{y}$$
$$x_3 = \dot{x}_2 = \ddot{y}$$

Substituting, we then find

$$A\dot{x}_3 + Bx_3 + Cx_2 + Dx_1 = f(t)$$

Finally, we get

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = \frac{1}{A}[f(t) - Bx_3 - Cx_2 - Dx_1] \end{cases}$$

Steady-State Frequency Response

- Sinusoidal input

$$f(t) = F_o \sin(\omega t)$$

- We assume that our system is asymptotically stable, we can calculate

$$G(iw) = G_R + iG_I$$

and

$$|G(iw)| = \sqrt{G_R^2 + G_I^2}$$

Steady-State Frequency Response Cont'd

- We need a quadrant check for the phase

$$\phi = \begin{cases} \tan^{-1}\left(\dfrac{G_I}{G_R}\right), & G_R \geq 0 \\ \tan^{-1}\left(\dfrac{G_I}{G_R}\right) + \pi, & G_R < 0 \end{cases}$$

- Putting it all together, we have

$$x_s(t) = F_o|G(i\omega)|\sin(\omega t + \phi)$$

Bode Plots

- A Bode plot is a plot of $20\ log|G|$ and ϕ vs. ω
- In class we discussed a procedure for sketching asymptotic Bode plots (we'll handle special cases separately)

1. Convert $G(s)$ from the form

$$G(s) = \frac{K(s+z_1)(s+z_2)\ \cdots\ (s+z_m)}{(s+p_1)(s+p_2)\ \cdots\ (s+z_n)}$$

to the form

$$G(s) = \frac{K'(1+\tau_1' s)(1+\tau_2' s)\cdots(1+\tau_m' s)}{(1+\tau_1 s)(1+\tau_2 s)\cdots(1+\tau_n s)}$$

Bode Plots Cont'd

- Procedure Cont'd

2. Calculate the corner frequencies
3. Begin the Bode plots one decade below the lowest corner frequency. The starting amplitude is $20\ log|K'|$ with a starting slope of 0 dB/dec.
4. At each corner frequency, the slope of the amplitude will change +20 dB/dec if the corner frequency is in the numerator and -20 dB/dec if the corner frequency is in the denominator.
5. At each corner frequency, the magnitude of the phase will jump +90 deg if the corner frequency is in the numerator and -90 deg if the corner frequency is in the denominator
6. Continue plotting until you are at least one decade above the highest corner frequency

Bode Plots Cont'd

- Procedure changes for special cases
  - Separate factor of s in the denominator of *G*

$$G(s) = \frac{K'(1+\tau_1' s)(1+\tau_2' s)\cdots(1+\tau_m' s)}{s^p(1+\tau_1 s)(1+\tau_2 s)\cdots(1+\tau_n s)}$$

Then start the amplitude plot with initial value $20\ \log\left(\frac{K'}{\omega_o^p}\right)$ and with an initial slope of -20**p* dB/dec. The initial phase will be 90**p* deg.

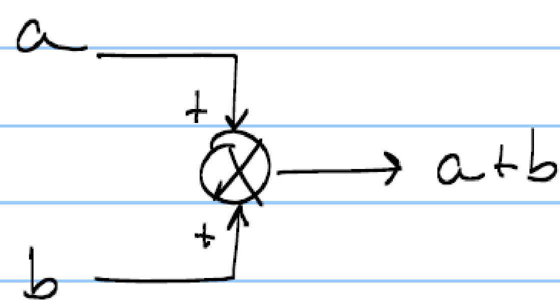
  - Quadratic terms

If quadratic terms appear, the slope of the amplitude changes by $\pm$40 dB/dec and the phase will shift by $\pm$180 deg.

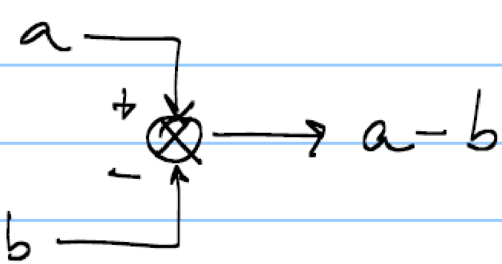
Block Diagrams and Block Algebra

- Always reduce your answers as much as time allows!
- Steady state error calculations (error is a function of system type and input)

- summing junction

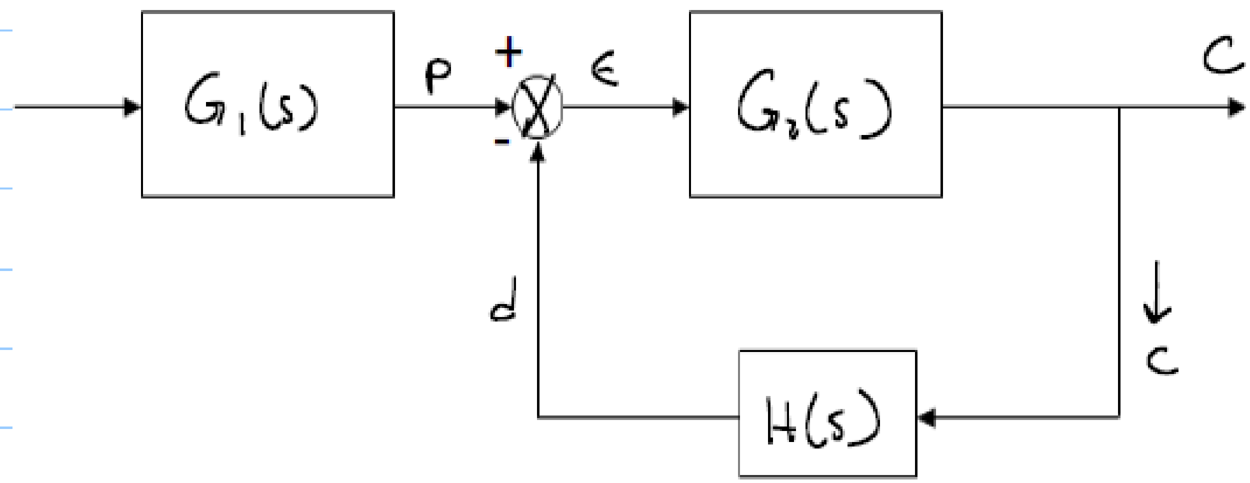


- negative summation



Example

Closed-loop transfer function



$p = G_1 r$

$\epsilon = p - d$

$c = G_2 \epsilon$

$d = H c$

want $C = [\quad\quad] r$

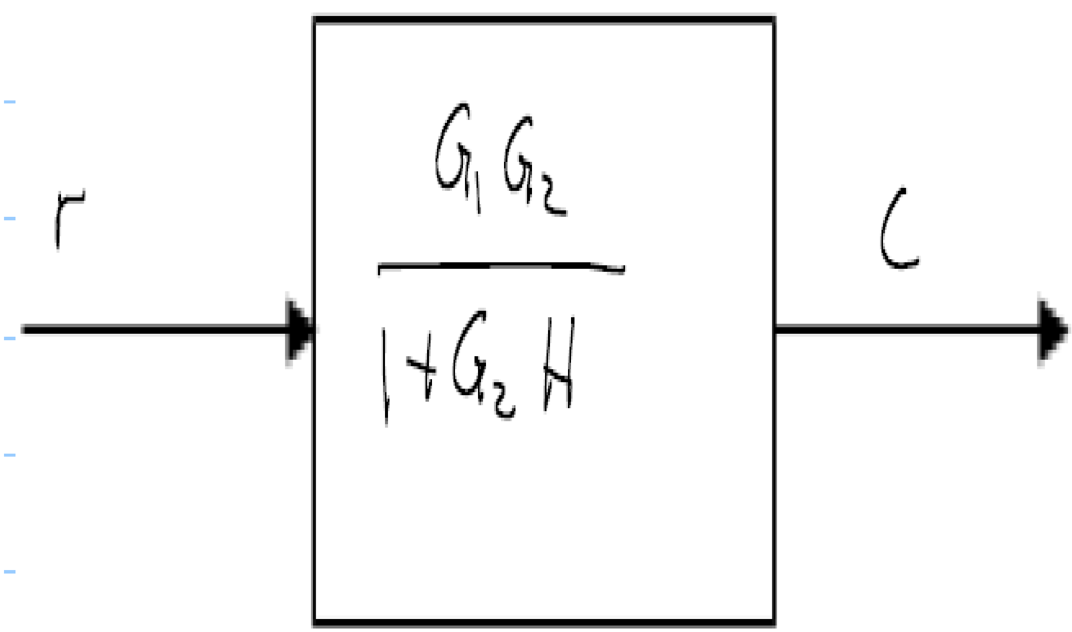
$c = G_2(p - d) = G_2(G_1 r - Hc)$

$c = G_2 G_1 r - G_2 H c$

$c(1 + G_2 H) = G_1 G_2 r$

$$c = \left[\frac{G_1 G_2}{1 + G_2 H}\right] r$$

↑ open loop transfer function



Controller considerations

- P = proportional, D = derivative, I = integral
- Various Combinations: PI controller, PD controller, PID controller, PID controller with filtering for the derivative action, …
- Tuning controllers is a bit of an inexact science (more on this in more advanced courses – for example, the Ziegler–Nichols tuning method)
- Routh tables are one approach, there are other methods you'll learn in more advanced courses (Root Locus, other MATLAB tools, etc.)

That’s all, folks!

- Thanks for being a good class this year ☺