Applications to Estimation & Filtering

Puneet Singla, Manoranjan Majji
Department of Aerospace Engineering
Pennsylvania State University, University Park, PA-16802
Texas A&M University, College Station, TX-77843

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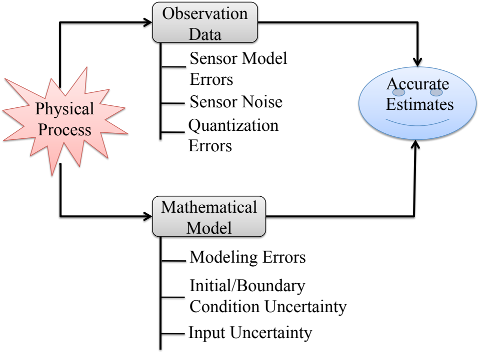


- The fusion of observational data with numerical simulation promises to provide greater understanding of physical phenomenon than either approach alone can achieve.
- The most critical challenge here is to provide a quantitative assessment of how closely our estimates reflect reality in the presence of model uncertainty as well as measurement errors and uncertainty.

Ideal Filter

Discrete Dynamics , Discrete Measurements

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \nu_k$$
$$\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{x}_{k+1}) + \omega_{k+1}$$

Continuous Dynamics , Discrete Measurements

$$d\mathbf{x} = f(\mathbf{x}_t, t)dt + d\beta_t \quad [t_k, t_{k+1}]$$
$$\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{x}_{k+1}) + \omega_{k+1}$$

IDEAL FILTER

$p(x_0)$

$$p(x_{k+1}|Y_k) = \int p(x_{k+1}|x_k)p(x_k|Y_k)dx_k$$

Discrete

$$\frac{\partial p}{\partial t} = -\sum \frac{\partial p f_i}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 [pQ_{ij}]}{\partial x_i \partial x_j}$$

Continuous
$[t_k, t_{k+1}]$

Discrete Time Measurement Update
Bayes' Rule

$$p(x_{k+1}|Y_{k+1}) = \frac{p(x_{k+1}|y_{k+1})p(x_{k+1}|Y_k)}{\int p(x_{k+1}|y_{k+1})p(x_{k+1}|Y_k)dx_{k+1}}$$

y_{k+1}

Assumptions

- Initial conditions, Process noise and Measurement noise are Gaussian distributed and mutually independent processes.
- State and Measurement model equations are linear.

Process and Measurement model

$$
\begin{aligned}
\mathbf{x}_{k+1} &= F_k \mathbf{x}_k + \nu_k \\
\mathbf{y}_{k+1} &= H_{k+1} \mathbf{x}_{k+1} + \omega_{k+1}
\end{aligned}
$$

where

- $\mathbf{x}_0 \sim \mathcal{N}(\mathbf{x}_0 : \hat{\mathbf{x}}_0, P_0)$
- $\nu_k \sim \mathcal{N}(\nu_k : \mathbf{0}, Q_k)$
- $\omega_{k+1} \sim \mathcal{N}(\omega_{k+1} : 0, R_{k+1})$

TIME EVOLUTION

Chapman-Kolmogorov Equation: Given the state pdf at time step k

$$p(\mathbf{x}_k|Y_k) = \mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k|k}, P_{k|k})$$

where Y_k is set of measurement upto time k,

$$p(\mathbf{x}_{k+1}|Y_k) = \int p(\mathbf{x}_{k+1}|\mathbf{x}_k) \, . \, p(\mathbf{x}_k|Y_k) \, d\mathbf{x}_k$$

The state transition probability density $p(\mathbf{x}_{k+1}|\mathbf{x}_k)$ is given as

$$p(\mathbf{x}_{k+1}|\mathbf{x}_k) = \mathcal{N}(\mathbf{x}_{k+1} : F_k\mathbf{x}_k, Q_k)$$

$$p(\mathbf{x}_{k+1}|Y_k) = \int \mathcal{N}(\mathbf{x}_{k+1} : F_k\mathbf{x}_k, Q_k) \, . \, \mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k|k}, P_{k|k}) \, d\mathbf{x}_k$$

Time Evolution

$$p(\mathbf{x}_{k+1}|Y_k) = \frac{1}{\sqrt{(2\pi)^n |P_{k+1|k}|}} . \exp\left[-\frac{1}{2}\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)^T P^{-1}_{k+1|k} \left(\mathbf{x}_{k+1}\hat{\mathbf{x}}_{k+1|k}\right)\right]$$

The prior pdf $p(\mathbf{x}_{k+1}|Y_k)$ has mean $\hat{\mathbf{x}}_{k+1|k}$ and covariance $P_{k+1|k}$ given as

$$\hat{\mathbf{x}}_{k+1|k} = F_k \hat{\mathbf{x}}_{k|k}$$
$$P_{k+1|k} = F_k P_{k|k} F^T + Q_k$$

Hence, the prior pdf remains Gaussian

As we know the pdf remains Gaussian, an easier approach is to directly compute the mean and covariance from the linear model equations as:

$$\hat{\mathbf{x}}_{k+1|k} = E[\mathbf{x}_{k+1}] = E\left[F_k x_k\right] + E[\nu_k] = F_k E\left[x_k\right] = F_k \hat{\mathbf{x}}_{k|k}$$
$$P_{k+1|k} = E\left[\left(\mathbf{x}_{k+1} - F_k\hat{\mathbf{x}}_{k|k}\right)\left(\mathbf{x}_{k+1} - F_k\hat{\mathbf{x}}_{k|k}\right)^T\right] = E\left[\left(F_k x_k + \nu_k - F_k\hat{\mathbf{x}}_{k|k}\right)\left(F_k x_k + \nu_k - F_k\hat{\mathbf{x}}_{k|k}\right)^T\right]$$
$$= E\left[F_k(x_k - \hat{\mathbf{x}}_{k|k})(x_k - \hat{\mathbf{x}}_{k|k})^T F_k^T + \nu_k \nu_k^T\right] = F_k P_{k|k} F^T + Q_k$$

Measurement update

Starting with the prior pdf at time step $k+1$,

$$p(\mathbf{x}_{k+1}|Y_k) = \mathcal{N}(\mathbf{x}_{k+1} : \hat{\mathbf{x}}_{k+1|k}, P_{k+1|k})$$

the posterior pdf from Bayes' rule at the same time step is given by

$$p(\mathbf{x}_{k+1}|Y_{k+1}) = \frac{p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})p(\mathbf{x}_{k+1}|Y_k)}{\int p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})p(\mathbf{x}_{k+1}|Y_k)\, d\mathbf{x}_{k+1}}$$

where the measurement likelihood pdf $p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})$ can be derived from the measurement model equations as:

$$p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1}) = \mathcal{N}(\mathbf{y}_{k+1} : H_{k+1}\mathbf{x}_{k+1}, R_{k+1})$$

Numerator of Bayes' rule is simplified as:

$$p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})p(\mathbf{x}_{k+1}|Y_k) = \mathcal{N}(\mathbf{y}_{k+1} : H_{k+1}\mathbf{x}_{k+1},\ R_{k+1})\mathcal{N}(\mathbf{x}_{k+1} : \hat{\mathbf{x}}_{k+1|k}\ , P_{k+1|k})$$

$$= \frac{1}{\sqrt{|2\pi R_{k+1}|}} \cdot \frac{1}{\sqrt{|2\pi P_{k+1|k}|}} \cdot \exp\Big[-\frac{1}{2}\left(\mathbf{y}_{k+1} - H_{k+1}\mathbf{x}_{k+1}\right)^T R_{k+1}^{-1}\left(\mathbf{y}_{k+1} - H_{k+1}\mathbf{x}_{k+1}\right) - \frac{1}{2}\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)^T P_{k+1|k}^{-1}\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)\Big]$$

The exponent is:

$$\Rightarrow -\frac{1}{2}\left(\mathbf{y}_{k+1} - H_{k+1}\mathbf{x}_{k+1}\right)^T R_{k+1}^{-1}\left(\mathbf{y}_{k+1} - H_{k+1}\mathbf{x}_{k+1}\right) - \frac{1}{2}\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)^T P_{k+1|k}^{-1}\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)$$

$$\Rightarrow -\frac{1}{2}\Big[\mathbf{y}_{k+1}^T R_{k+1}^{-1}\mathbf{y}_{k+1} + \mathbf{x}_{k+1}^T \underbrace{\left(P_{k+1|k}^{-1} + H_{k+1}^T R_{k+1}^{-1} H_{k+1}\right)}_{A}\mathbf{x}_{k+1} - 2\underbrace{\left(\mathbf{y}_{k+1}^T R_{k+1}^{-1} H_{k+1} + \hat{\mathbf{x}}_{k+1|k}^T P_{k+1|k}^{-1}\right)}_{b^T}\mathbf{x}_{k+1} + \hat{\mathbf{x}}_{k+1|k}^T P_{k+1|k}^{-1}\hat{\mathbf{x}}_{k+1|k}\Big]$$

$$\Rightarrow -\frac{1}{2}\left[\mathbf{y}_{k+1}^T R_{k+1}^{-1}\mathbf{y}_{k+1} + \mathbf{x}_{k+1}^T A\mathbf{x}_{k+1} - 2b^T\mathbf{x}_{k+1} + \hat{\mathbf{x}}_{k+1|k}^T P_{k+1|k}^{-1}\hat{\mathbf{x}}_{k+1|k}\right]$$

The denominator of Bayes' rule is given as:

$$\int p(\mathbf{y}_{k+1}|\mathbf{x}_{k+1})p(\mathbf{x}_{k+1}|Y_k)\,d\mathbf{x}_{k+1} = \frac{1}{\sqrt{|2\pi R_{k+1}|}}\cdot\frac{1}{\sqrt{|2\pi P_{k+1|k}|}}\cdot$$

$$\int \exp\left[-\frac{1}{2}\mathbf{y}_{k+1}^T R_{k+1}^{-1}\mathbf{y}_{k+1} - \frac{1}{2}\mathbf{x}_{k+1}^T A\mathbf{x}_{k+1} + b^T\mathbf{x}_{k+1} - \frac{1}{2}\hat{\mathbf{x}}_{k+1|k}^T P_{k+1|k}^{-1}\hat{\mathbf{x}}_{k+1|k}\right] d\mathbf{x}_{k+1}$$

$$= \frac{1}{\sqrt{|2\pi R_{k+1}|}}\cdot\frac{1}{\sqrt{|2\pi P_{k+1|k}|}}\cdot\exp\left[-\frac{1}{2}\mathbf{y}_{k+1}^T R_{k+1}^{-1}\mathbf{y}_{k+1} - \frac{1}{2}\hat{\mathbf{x}}_{k+1|k}^T P_{k+1|k}^{-1}\hat{\mathbf{x}}_{k+1|k}\right]$$

$$\cdot\int \exp\left[-\frac{1}{2}\mathbf{x}_{k+1}^T A\mathbf{x}_{k+1} + b^T\mathbf{x}_{k+1}\right] d\mathbf{x}_{k+1}$$

$$= \frac{\sqrt{|2\pi A^{-1}|}}{\sqrt{|2\pi R_{k+1}||2\pi P_{k+1|k}|}}\cdot\exp\left[-\frac{1}{2}\mathbf{y}_{k+1}^T R_{k+1}^{-1}\mathbf{y}_{k+1} - \frac{1}{2}\hat{\mathbf{x}}_{k+1|k}^T P_{k+1|k}^{-1}\hat{\mathbf{x}}_{k+1|k} + \frac{1}{2}b^T A^{-T} b\right]$$

The posterior pdf from Bayes' rule is given as:

$$p(\mathbf{x}_{k+1}|Y_k) = \frac{1}{\sqrt{|2\pi A^{-1}|}}\cdot\exp\left[-\frac{1}{2}\mathbf{x}_{k+1}^T A\mathbf{x}_{k+1} + b^T\mathbf{x}_{k+1} - \frac{1}{2}b^T A^{-T} b\right]$$

$$= \frac{1}{\sqrt{|2\pi A^{-1}|}}\cdot\exp\left[-\frac{1}{2}\left(\mathbf{x}_{k+1} - A^{-1}b\right)^T A\left(\mathbf{x}_{k+1} - A^{-1}b\right)\right]$$

$$p(\mathbf{x}_{k+1}|Y_{k+1}) = \frac{1}{\sqrt{|2\pi A^{-1}|}} \cdot \exp\left[-\frac{1}{2}\left(\mathbf{x}_{k+1} - A^{-1}b\right)^T A\left(\mathbf{x}_{k+1} - A^{-1}b\right)\right]$$

$$\begin{aligned}
b^T &= \left(\mathbf{y}_{k+1}^T R_{k+1}^{-1} H_{k+1} + \hat{\mathbf{x}}_{k+1|k}^T P_{k+1|k}^{-1}\right) \\
A &= P_{k+1|k}^{-1} + H_{k+1}^T R_{k+1}^{-1} H_{k+1} \\
A^{-1} &= P_{k+1|k} - \underbrace{P_{k+1|k} H_{k+1}^T \left(R_{k+1} + H_{k+1} P_{k+1|k} H_{k+1}^T\right)^{-1}}_{K_{k+1}} H_{k+1} P_{k+1|k} \\
&= P_{k+1|k} - K_{k+1} H_{k+1} P_{k+1|k} \\
A^{-1}b &= \left(P_{k+1|k} - K_{k+1} H_{k+1} P_{k+1|k}\right)\left(H_{k+1}^T R_{k+1}^{-1} \mathbf{y}_{k+1} + P_{k+1|k}^{-1} \hat{\mathbf{x}}_{k+1|k}\right) \\
&= \hat{\mathbf{x}}_{k+1|k} + K_{k+1}\left(\mathbf{y}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{k+1|k}\right)
\end{aligned}$$

Hence the posterior pdf still remains Gaussian even after Bayes' Rule update, with mean and covariance as

$$\begin{aligned}
\hat{\mathbf{x}}_{k+1|k+1} &= A^{-1}b = \hat{\mathbf{x}}_{k+1|k} + K_{k+1}\left(\mathbf{y}_{k+1} - H_{k+1} \hat{\mathbf{x}}_{k+1|k}\right) \\
P_{k+1|k+1} &= A^{-1} = P_{k+1|k} - K_{k+1} H_{k+1} P_{k+1|k}
\end{aligned}$$

In summary, starting from time step k with $p(\mathbf{x}_k|Y_k) = \mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k|k}, P_{k|k})$

TIME EVOLUTION

$$p(\mathbf{x}_{k+1}|Y_k) = \mathcal{N}(\mathbf{x}_{k+1} : \hat{\mathbf{x}}_{k+1|k}, P_{k+1|k})$$
$$\hat{\mathbf{x}}_{k+1|k} = F_k \hat{\mathbf{x}}_{k|k}$$
$$P_{k+1|k} = F_k P_{k|k} F^T + Q_k$$

MEASUREMENT UPDATE

$$p(\mathbf{x}_{k+1}|Y_{k+1}) = \mathcal{N}(\mathbf{x}_{k+1} : \hat{\mathbf{x}}_{k+1|k+1}, P_{k+1|k+1})$$
$$\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + K_{k+1}\left(\mathbf{y}_{k+1} - H_{k+1}\hat{\mathbf{x}}_{k+1|k}\right)$$
$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1} H_{k+1} P_{k+1|k}$$
$$K_{k+1} = P_{k+1|k} H_{k+1}^T \left(R_{k+1} + H_{k+1} P_{k+1|k} H_{k+1}^T\right)^{-1}$$

The Kalman filter fuses the system dynamic model with measurement data in an optimal manner.

Minimum Variance Estimate

$$\min_{\hat{\mathbf{x}}_{k+1/k+1}} Tr\left\{E[(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k+1})(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k+1})^T]\right\}$$

i.e. find an estimate that minimizes the posterior variance

Estimator: Kalman Filter like update

$$\hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})$$

The assumed estimator is unbiased:

$$\begin{aligned} E[\hat{\mathbf{x}}_{k+1/k+1}] &= E[\hat{\mathbf{x}}_{k+1/k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})] = E[\hat{\mathbf{x}}_{k+1/k}] + \mathbf{K}_{k+1}(E[\mathbf{y}_{k+1}] - E[\hat{\mathbf{y}}_{k+1}]) \\ &= F_k E[\hat{\mathbf{x}}_{k|k}] + \mathbf{K}_{k+1}(H_{k+1}E[\mathbf{x}_{k+1}] - H_{k+1}E[\hat{\mathbf{x}}_{k+1/k}]) \\ &= E[\mathbf{x}_{k+1}] + \mathbf{K}_{k+1}(H_{k+1}F\hat{\mathbf{x}}_{k|k} - H_{k+1}F\hat{\mathbf{x}}_{k|k}) = E[\mathbf{x}_{k+1}] \end{aligned}$$

$$\min_{\mathbf{K}_{k+1}} Tr\Big\{E[(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k+1})(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k+1})^T]\Big\}$$
$$with \quad \hat{\mathbf{x}}_{k+1/k+1}=\hat{\mathbf{x}}_{k+1/k}+\mathbf{K}_{k+1}(\mathbf{y}_{k+1}-\hat{\mathbf{y}}_{k+1})$$

$$\min_{\mathbf{K}_{k+1}} Tr\Big\{E[(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k}-\mathbf{K}_{k+1}(\mathbf{y}_{k+1}-\hat{\mathbf{y}}_{k+1}))(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k}-\mathbf{K}_{k+1}(\mathbf{y}_{k+1}-\hat{\mathbf{y}}_{k+1}))^T]\Big\}$$
$$\min_{\mathbf{K}_{k+1}} Tr\Big\{E[\big(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k}\big)\big(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k}\big)^T]+\mathbf{K}_{k+1}E[(\mathbf{y}_{k+1}-\hat{\mathbf{y}}_{k+1})(\mathbf{y}_{k+1}-\hat{\mathbf{y}}_{k+1})^T]\mathbf{K}_{k+1}^T$$
$$-\mathbf{K}_{k+1}E[(\mathbf{y}_{k+1}-\hat{\mathbf{y}}_{k+1})(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k})^T]-E[\big(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k}\big)\big(\mathbf{y}_{k+1}-\hat{\mathbf{y}}_{k+1}\big)^T]\mathbf{K}_{k+1}^T\Big\}$$
$$\min_{\mathbf{K}_{k+1}} Tr\Big\{P_{k+1|k}+\mathbf{K}_{k+1}P^{y}_{k+1|k}\mathbf{K}_{k+1}^T-\mathbf{K}_{k+1}P^{yx}_{k+1|k}-P^{xy}_{k+1|k}\mathbf{K}_{k+1}^T\Big\}$$

The optimal gain K_{k+1} is given by

$$\mathbf{K}_{k+1}=\mathbf{P}^{xy}_{k+1}(\mathbf{P}^{y}_{k+1/k})^{-1}$$

- no assumptions on the state pdf.
- All the expectations are with respect to the prior pdf $p(\mathbf{x}_{k+1}|Y_k)$.

$$
\begin{aligned}
\hat{\mathbf{y}}_{k+1} &= E[\mathbf{y}_{k+1}] = E[H_{k+1}x_{k+1} + \omega_{k+1}] = H_{k+1}E[x_{k+1}] = H_{k+1}\hat{\mathbf{x}}_{k+1|k} \\
P^{y}_{k+1/k} &= E[(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})^T] \\
&= E[\left(H_{k+1}\mathbf{x}_{k+1} + \omega_{k+1} - H_{k+1}\hat{\mathbf{x}}_{k+1|k}\right)\left(H_{k+1}\mathbf{x}_{k+1} + \omega_{k+1} - H_{k+1}\hat{\mathbf{x}}_{k+1|k}\right)^T] \\
&= H_{k+1}E[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)^T]H^T_{k+1} + E[\omega_{k+1}\omega^T_{k+1}] \\
&= H_{k+1}P_{k+1|k}H^T_{k+1} + R_{k+1} \\
P^{xy}_{k+1} &= E[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k}\right)(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})^T] \\
&= E[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k}\right)\left(H_{k+1}(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k}) + \omega_{k+1}\right)^T] \\
&= E[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)^T]H_{k+1} = P_{k+1|k}H^T_{k+1}
\end{aligned}
$$

The Kalman Filter gain is then given as:

$$
\mathbf{K}_{k+1} = \mathbf{P}^{xy}_{k+1}(\mathbf{P}^{y}_{k+1/k})^{-1} = P_{k+1|k}H^T_{k+1}\left(H_{k+1}P_{k+1|k}H^T_{k+1} + R_{k+1}\right)^{-1}
$$

Minimum variance estimator is same as Kalman Filter and hence optimal for linear system with Gaussian pdfs

Extended Kalman Filter: Linearization of system model equations

PROCESS AND MEASUREMENT MODEL

$$\mathbf{x}_{k+1} = f_k(\mathbf{x}_k) + \nu_k$$
$$\mathbf{y}_{k+1} = h_{k+1}(\mathbf{x}_{k+1}) + \omega_{k+1}$$

Starting with the most recent estimates at time k i.e. $\hat{\mathbf{x}}_{k|k}$ and $P_{k|k}$, the posterior estimates are approximated as

$$\begin{aligned} E[\mathbf{x}_{k+1}] &= E[f_k(\mathbf{x}_k) + \nu_k] = E[f_k(\mathbf{x}_k)] + E[\nu_k] = E[f_k(\mathbf{x}_k)] \\ &= E[f_k(\mathbf{x}_k)] = E[f_k(\hat{\mathbf{x}}_{k|k}) + F_k\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right) + \ldots] \approx f_k(\hat{\mathbf{x}}_{k|k}) \end{aligned}$$

using the approximation

$$f_k(\mathbf{x}_k) - f_k(\hat{\mathbf{x}}_{k|k}) \approx F_k\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right) \quad where \quad F_k \equiv \frac{\partial f_k}{\partial \mathbf{x}_k}|_{\mathbf{x}_k=\hat{\mathbf{x}}_{k|k}}$$

$$\begin{aligned}
E[&\left(\mathbf{x}_{k+1} - f_k(\hat{\mathbf{x}}_{k|k})\right)\left(\mathbf{x}_{k+1} - f_k(\hat{\mathbf{x}}_{k|k})\right)^T] \\
&= E[\left(f_k(\mathbf{x}_k) + \nu_k - f_k(\hat{\mathbf{x}}_{k|k})\right)\left(f_k(\mathbf{x}_k) + \nu_k - f_k(\hat{\mathbf{x}}_{k|k})\right)^T] \\
&= E[\left(F_k\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right) + \nu_k\right)\left(F_k\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right) + \nu_k\right)^T] \\
&= F_k E[\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right)\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right)^T]F_k^T + E[\nu_k \nu_k^T] = F_k P_{k|k} F_k^T + Q_k
\end{aligned}$$

Only mean and covariance are propagated

EKF: TIME EVOLUTION STEP

$$\hat{\mathbf{x}}_{k+1|k} = f_k(\hat{\mathbf{x}}_{k|k})$$

$$P_{k+1|k} = F_k P_{k|k} F_k^T + Q_k$$

Minimum variance estimator with Kalman Filter like update for the EKF:

$$\min_{\mathbf{K}_{k+1}} Tr\left\{E[(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k+1})(\mathbf{x}_{k+1}-\hat{\mathbf{x}}_{k+1/k+1})^T]\right\}$$

$$with \quad \hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})$$

$$\min_{\mathbf{K}_{k+1}} Tr\left\{P_{k+1|k} + \mathbf{K}_{k+1}P^{y}_{k+1|k}\mathbf{K}^T_{k+1} - \mathbf{K}_{k+1}P^{yx}_{k+1|k} - P^{xy}_{k+1|k}\mathbf{K}^T_{k+1}\right\}$$

The optimal gain K_{k+1} is given by

$$\mathbf{K}_{k+1} = \mathbf{P}^{xy}_{k+1}(\mathbf{P}^{y}_{k+1/k})^{-1}$$

Using Taylor Series Expansion of $h(x_{k+1})$ about the current estimate at time $k+1$ i.e. $\hat{\mathbf{x}}_{k+1|k}$

$$h_{k+1}(\mathbf{x}_{k+1}) = h_{k+1}(\hat{\mathbf{x}}_{k+1|k}) + H_{k+1}(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}) \quad where \quad H_{k+1} \equiv \frac{\partial h_{k+1}}{\partial \mathbf{x}_{k+1}}\Big|_{\mathbf{x}_{k+1}=\hat{\mathbf{x}}_{k+1|k}}$$

$$\hat{\mathbf{y}}_{k+1} = E[\mathbf{y}_{k+1}] = E[h_{k+1}(\mathbf{x}_{k+1})] + E[\omega_{k+1}] \approx h_{k+1}\left(\hat{\mathbf{x}}_{k+1|k}\right)$$

$$
\begin{aligned}
P^{y}_{k+1/k} &= E[(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})^T] \\
&\approx E\left[\left(h_{k+1}(\mathbf{x}_{k+1}) + \omega_{k+1} - h_{k+1}(\hat{\mathbf{x}}_{k+1|k})\right)\left(h_{k+1}(\mathbf{x}_{k+1}) + \omega_{k+1} - h_{k+1}(\hat{\mathbf{x}}_{k+1|k})\right)^T\right] \\
&= H_{k+1} E\left[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)^T\right] H^T_{k+1} + E[\omega_{k+1}\omega^T_{k+1}] \\
&= H_{k+1} P_{k+1|k} H^T_{k+1} + R_{k+1} \\
P^{xy}_{k+1} &= E\left[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k}\right)(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})^T\right] \\
&\approx E\left[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k}\right)\left(h_{k+1}(\mathbf{x}_{k+1}) + \omega_{k+1} - h_{k+1}(\hat{\mathbf{x}}_{k+1|k})\right)^T\right] \\
&= E\left[\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)\left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k}\right)^T\right] H_{k+1} = P_{k+1|k} H^T_{k+1}
\end{aligned}
$$

The Extended Kalman Filter gain is then given as

$$
\mathbf{K}_{k+1} = \mathbf{P}^{xy}_{k+1}(\mathbf{P}^{y}_{k+1/k})^{-1} = P_{k+1|k} H^T_{k+1}\left(H_{k+1} P_{k+1|k} H^T_{k+1} + R_{k+1}\right)^{-1}
$$

In summary,

Time Evolution

$$\hat{\mathbf{x}}_{k+1|k} = f_k(\hat{\mathbf{x}}_{k|k})$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + Q_k$$
$$F_k \equiv \frac{\partial f_k}{\partial \mathbf{x}_k}\Big|_{\mathbf{x}_k=\hat{\mathbf{x}}_{k|k}}$$

Measurement Update

$$\hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - h_{k+1}(\hat{\mathbf{x}}_{k+1|k}))$$
$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1} H_{k+1} P_{k+1|k}$$
$$\mathbf{K}_{k+1} = P_{k+1|k} H_{k+1}^T \left(H_{k+1} P_{k+1|k} H_{k+1}^T + R_{k+1}\right)^{-1}$$
$$H_{k+1} \equiv \frac{\partial h_{k+1}}{\partial \mathbf{x}_{k+1}}\Big|_{\mathbf{x}_{k+1}=\hat{\mathbf{x}}_{k+1|k}}$$

- Only mean and covariance are propagated and updated.
- All expectation expression $E[.]$ evaluated by linearizations $\Rightarrow$ analytical expressions
- Estimates can quickly diverge due to linearizations involved.

Motivation for Unscented and Quadrature Kalman Filter

- $\Rightarrow$ Avoid linearization altogether and evaluate these expectation integrals directly using appropriate quadrature scheme.
- *Unscented Transform, Gauss-hermite Quadratures or Conjugate Unscented Transform*

Discrete Dynamic System

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) + \nu_k$$
$$\mathbf{y}_{k+1} = \mathbf{h}(\mathbf{x}_{k+1}) + \omega_{k+1}$$

Evolution of the exact conditional pdf is given by two stages

Time Evolution step

$$p(\mathbf{x}_{k+1}|\mathbf{Y}_k) = \int \mathscr{N}(\mathbf{x}_{k+1} : \mathbf{f}(\mathbf{x}_k), \mathbf{Q}_k) \,.\, p(\mathbf{x}_k|\mathbf{Y}_k) d\mathbf{x}_k$$

Measurement Update step

$$p(\mathbf{x}_{k+1}|\mathbf{Y}_{k+1}) = \frac{\mathscr{N}(\mathbf{y}_{k+1} : \mathbf{h}(\mathbf{x}_{k+1}), \mathbf{R}_{k+1}) \,.\, p(\mathbf{x}_{k+1}|\mathbf{Y}_k)}{\int \mathscr{N}(\mathbf{y}_{k+1} : \mathbf{h}(\mathbf{x}_{k+1}), \mathbf{R}_{k+1}) \,.\, p(\mathbf{x}_{k+1}|\mathbf{Y}_k)\, d\mathbf{x}_{k+1}}$$

Gaussian Approximated conditional pdfs:

TIME EVOLUTION STEP

$$p(\mathbf{x}_{k+1}|\mathbf{Y}_k) = \int \mathcal{N}(\mathbf{x}_{k+1} : \mathbf{f}(\mathbf{x}_k), \mathbf{Q}_k).p(\mathbf{x}_k|\mathbf{Y}_k)d\mathbf{x}_k$$

$$Mean : \int \mathbf{x}_{k+1} p(\mathbf{x}_{k+1}|\mathbf{Y}_k)\, d\mathbf{x}_{k+1} \Rightarrow \hat{\mathbf{x}}_{k+1/k}$$

$$Covariance : \int (\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k})^T p(\mathbf{x}_{k+1}|\mathbf{Y}_k)\, d\mathbf{x}_{k+1} \Rightarrow \mathbf{P}_{k+1/k}$$

$$\hat{\mathbf{x}}_{k+1/k} = \int \mathbf{f}(\mathbf{x}_k)\mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k/k}, \mathbf{P}_{k/k}|\mathbf{Y}_k)d\mathbf{x}_k$$

$$\mathbf{P}_{k+1/k} = \int \mathbf{f}(\mathbf{x}_k)\mathbf{f}(\mathbf{x}_k)^T \mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k/k}, \mathbf{P}_{k/k}|\mathbf{Y}_k)d\mathbf{x}_k - \hat{\mathbf{x}}_{k+1/k}\hat{\mathbf{x}}^T_{k+1/k} + \mathbf{Q}_k$$

GAUSSIAN PDF AT $k+1$

$$p(\mathbf{x}_{k+1}|\mathbf{Y}_k) \approx \mathcal{N}(\mathbf{x}_{k+1} : \hat{\mathbf{x}}_{k+1/k}, \mathbf{P}_{k+1/k})$$

Gaussian Approximated conditional pdfs:

TIME EVOLUTION STEP

$$p(\mathbf{x}_{k+1}|\mathbf{Y}_k) = \int \mathcal{N}(\mathbf{x}_{k+1} : \mathbf{f}(\mathbf{x}_k), \mathbf{Q}_k).\mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k/k}, \mathbf{P}_{k/k}) d\mathbf{x}_k$$

$$Mean : \int \mathbf{x}_{k+1} p(\mathbf{x}_{k+1}|\mathbf{Y}_k)\, d\mathbf{x}_{k+1} \Rightarrow \hat{\mathbf{x}}_{k+1/k}$$

$$Covariance : \int (\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k})^T p(\mathbf{x}_{k+1}|\mathbf{Y}_k)\, d\mathbf{x}_{k+1} \Rightarrow \mathbf{P}_{k+1/k}$$

$$\hat{\mathbf{x}}_{k+1/k} = \int \mathbf{f}(\mathbf{x}_k) \mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k/k}, \mathbf{P}_{k/k}|\mathbf{Y}_k) d\mathbf{x}_k$$

$$\mathbf{P}_{k+1/k} = \int \mathbf{f}(\mathbf{x}_k)\mathbf{f}(\mathbf{x}_k)^T \mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k/k}, \mathbf{P}_{k/k}|\mathbf{Y}_k) d\mathbf{x}_k - \hat{\mathbf{x}}_{k+1/k}\hat{\mathbf{x}}_{k+1/k}^T + \mathbf{Q}_k$$

GAUSSIAN PDF AT $k+1$

$$p(\mathbf{x}_{k+1}|\mathbf{Y}_k) \approx \mathcal{N}(\mathbf{x}_{k+1} : \hat{\mathbf{x}}_{k+1/k}, \mathbf{P}_{k+1/k})$$

Assuming a Kalman Filter like update

$$\hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})$$

$$\min_{\mathbf{K}_{k+1}} Tr\left\{E[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k+1})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1/k+1})^T]\right\} \tag{1}$$

$$= Tr\left\{\mathbf{P}_{k+1/k}\right\} - Tr\left\{\mathbf{P}^{xy}_{k+1}\mathbf{K}^T_{k+1}\right\} - Tr\left\{\mathbf{K}_{k+1}(\mathbf{P}^{xy}_{k+1})^T\right\} + Tr\left\{\mathbf{K}_{k+1}\mathbf{P}^{y}_{k+1/k}\mathbf{K}^T_{k+1}\right\} \tag{2}$$

The Kalman gain is given as

$$\mathbf{K}_{k+1} = \mathbf{P}^{xy}_{k+1}(\mathbf{P}^{y}_{k+1/k})^{-1}$$

Kalman Filter like Update $p(\mathbf{x}_{k+1}|\mathbf{y}_{k+1})$

$$\hat{\mathbf{x}}_{k+1/k+1} = \hat{\mathbf{x}}_{k+1/k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1})$$

$$\mathbf{P}_{k+1/k+1} = \mathbf{P}_{k+1/k} - \mathbf{K}_{k+1}(\mathbf{P}^{xy}_{k+1/k})^T$$

$$\mathbf{K}_{k+1} = \mathbf{P}^{xy}_{k+1}(\mathbf{P}^{y}_{k+1/k})^{-1}$$

$$p(\mathbf{x}_{k+1}|\mathbf{Y}_{k+1}) \approx \mathcal{N}(\mathbf{x}_{k+1} : \hat{\mathbf{x}}_{k+1/k+1}, \mathbf{P}_{k+1/k+1})$$

The integrals to be evaluated are summarised as

TIME EVOLUTION STEP

$$\hat{\mathbf{x}}_{k+1/k} = \int \mathbf{f}(\mathbf{x}_k)\mathscr{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k/k}, \mathbf{P}_{k/k}|\mathbf{Y}_k)d\mathbf{x}_k$$

$$\mathbf{P}_{k+1/k} = \int \mathbf{f}(\mathbf{x}_k)\mathbf{f}(\mathbf{x}_k)^T\mathscr{N}(\mathbf{x}_k : \hat{\mathbf{x}}_{k/k}, \mathbf{P}_{k/k}|\mathbf{Y}_k)d\mathbf{x}_k - \hat{\mathbf{x}}_{k+1/k}\hat{\mathbf{x}}_{k+1/k}^T + \mathbf{Q}_k$$

MEASUREMENT UPDATE STEP

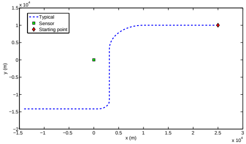
$$\hat{\mathbf{y}}_{k+1} = \int \mathbf{h}(\mathbf{x}_{k+1/k})\mathscr{N}(\mathbf{x}_{k+1/k} : \hat{\mathbf{x}}_{k+1/k}, \mathbf{P}_{k+1/k})d\mathbf{x}_{k+1/k}$$

$$\mathbf{P}^y_{k+1/k} = \int \mathbf{h}(\mathbf{x}_{k+1/k})\mathbf{h}(\mathbf{x}_{k+1/k})^T\mathscr{N}(\mathbf{x}_{k+1/k} : \hat{\mathbf{x}}_{k+1/k}, \mathbf{P}_{k+1/k})d\mathbf{x}_{k+1/k} - \hat{\mathbf{y}}_{k+1}\hat{\mathbf{y}}_{k+1}^T + \mathbf{R}_{k+1}$$

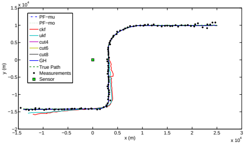
$$\mathbf{P}^{xy}_{k+1} = \int \mathbf{x}_{k+1/k}\mathbf{h}(\mathbf{x}_{k+1/k})^T\mathscr{N}(\mathbf{x}_{k+1/k} : \hat{\mathbf{x}}_{k+1/k}, \mathbf{P}_{k+1/k})d\mathbf{x}_{k+1/k} - \hat{\mathbf{x}}_{k+1/k}\hat{\mathbf{y}}_{k+1}^T$$

$(\mathbf{X};\mathbf{w}) \in$

- *Monte Carlo samples*
- *Gauss-Hermite Product rule*
- *Sparse Grid Gauss-Hermite Quadrature rule*
- *Unscented Transform*
- *Cubature Kalman Filter*
- *Conjugate Unscented Transform*
- *Minimal Cubature rules*



(a) Typical Aeroplane Trajectory



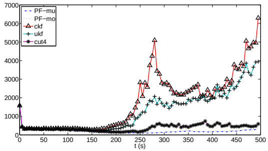
(b) Estimated Aeroplane Trajectory

Figure: *Air Traffic Scenario*

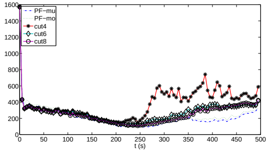
$$x_k = \begin{bmatrix} 1 & \frac{sin(\Omega T)}{\Omega} & 0 & -\frac{1-cos(\Omega T)}{\Omega} & 0 \\ 0 & cos(\Omega T) & 0 & -sin(\Omega T) & 0 \\ 0 & \frac{1-cos(\Omega T)}{\Omega} & 1 & \frac{sin(\Omega T)}{\Omega} & 0 \\ 0 & sin(\Omega T) & 0 & cos(\Omega T) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x_{k-1} + \nu_{k-1}, \quad \begin{bmatrix} r_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} \sqrt{(\xi_k)^2 + (\eta_k)^2} \\ tan^{-1}(\frac{\eta_k}{\xi_k}) \end{bmatrix} + \omega_k$$

Table: Comparison of 2-norms of RMSE in position, velocity and turn rate for $T = 5s$ for Air Traffic Problem.

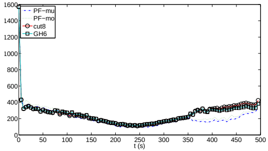
$\|\|RMSE\|\|_2$ in	$PF-mean$	CKF	UKF	$CUT4$	$CUT6$	$CUT8$
Position	115.47	989.15	685.90	245.30	138.82	135.89
Velocity	24.16	17330.23	12849.90	6127.86	2153.53	34.73
Ω	0.0393	2.873	2.396	1.329	0.636	0.090
No.of points	5000	10	11	42	83	355



(a) PF,CKF,UKF,CUT4

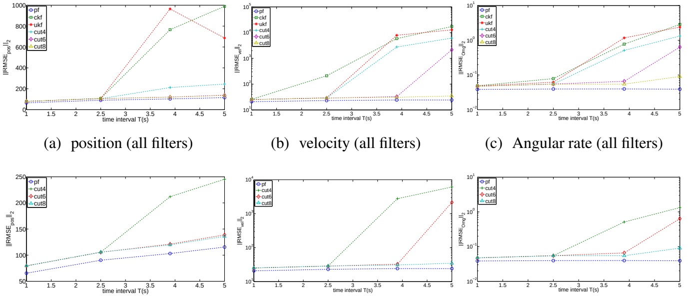


(b) PF,CUT4,CUT6,CUT8



(c) PF,CUT8,GH6

Figure: *RMSE in position for Air Traffic P*



(a) position (all filters) (b) velocity (all filters) (c) Angular rate (all filters)

(d) position (PF,CUT4,CUT6,CUT8) (e) velocity (PF,CUT4,CUT6,CUT8) (f) Angular rate (PF,CUT4,CUT6,CUT8)

Figure: *State Estimation Error vs. the Measurement Time Interval, T for the Air Traffic problem*

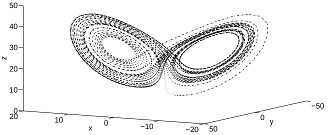
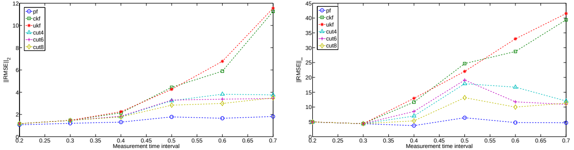


Figure: *Lorenz system*

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = xy - \beta z$$

$\sigma = 10$, $\rho = 28$ and $\beta = 8/3$. For uncertain σ and ρ, the appended state vector is $[x, y, z, \sigma, \rho]^T$ with mean and covarianc

$$\mu_0 = [1.50887, -1.531271, 25.46091, 10, 28]^T;$$
$$P_0 = Diag([4, 4, 4, 2, 4]^T)$$



(a) 2 norm of RMSE   (b) max of RMSE

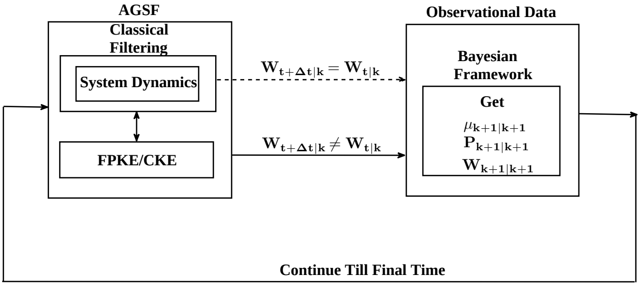
Figure: *Comparison of Filters for Lorenz model with varying measurement time intervals T*

TABLE: Comparison of RMSE for various filters

	$PF-mean$	CKF	UKF	$CUT4$	$CUT6$	$CUT8$
$\|\|RMSE\|\|_2$ *with* $Q = 0.002 I_{5\times5}$	14.7986	0.8945	0.8934	0.8974	0.9004	0.9002
$\|\|RMSE\|\|_2$ *with* $Q = 0.005 I_{5\times5}$	1.0826	6.3419	4.2319	1.9162	1.8156	1.7225
No. of pts	5000	10	11	42	83	355

Measurement Model:

$$\mathbf{z}_k = \mathbf{h}(t_k, \mathbf{x}_k) + \boldsymbol{\nu}_k \qquad \boldsymbol{\nu}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$$



- Use EKF or UKF measurement equations to get $\boldsymbol{\mu}_{k+1|k+1}$, $\mathbf{P}_{k+1|k+1}$.
- Use Bayes' rule to get the new weights.

Consider the following measurement model in discrete time

$$\mathbf{z}_k = \mathbf{h}(t_k, \mathbf{x}_k) + \boldsymbol{\nu}_k \qquad \boldsymbol{\nu}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k) \tag{3}$$

Then the Bayes' rule updates the weight according to the following equation:

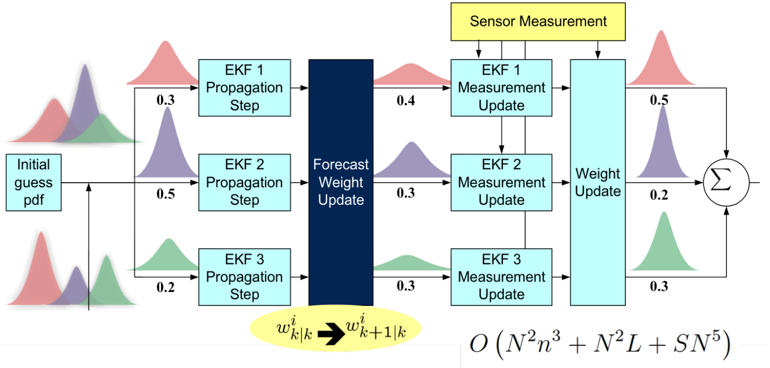
$$\mathbf{w}^i_{k+1|k+1} = \frac{w^i_{k+1|k}\gamma^i}{\sum_{i=1}^N w^i_{k+1|k}\gamma^i} \qquad i = 1, 2, ..., N \tag{4}$$

where $N =$ total number of Gaussian components and $\gamma^i \sim \mathcal{N}(\mathbf{z}_{k+1} - \boldsymbol{\mu}^i_{k+1|k}, \mathbf{H}^{(i)}_k \mathbf{P}^{(i)}_{k+1|k} \left(\mathbf{H}^{(i)}_k\right)^T + \mathbf{R}_k)$ for EKF and $\gamma^i \sim \mathcal{N}(\mathbf{z}_{k+1} - \boldsymbol{\mu}^i_{k+1|k}, \hat{S}_{k+1})$ for UKF.

$$\mathbf{H}_k = \frac{\partial}{\partial \mathbf{x}_k}\mathbf{h}(t_k, \mathbf{x}_k)|_{\mathbf{x}_k = \boldsymbol{\mu}_{k+1|k}} \tag{5}$$

$\hat{S}_{k+1} =$ innovation covariance[1]

[1] Julier, S., and Uhlmann, J.,"Unscented Filtering and Nonlinear Estimation"



- Highly Parallelized framework for Bayesian Nonlinear Filtering.

Measurements are available when the satellite is in the field of view of the radar of an observation center. For simulation purposes, we consider the observation center to be located at 39.007° latitude and 104.883° longitude near Air Force Academy in Colorado springs. The cartesian coordinates of this location is given as:

$$\mathbf{rSite} = \begin{bmatrix} -1275.1219 & -4797.9890 & 3994.2975 \end{bmatrix} \quad \text{km}$$

It is assumed that the measurements are available after 5.6585 hours i.e. time at which the last time-update was made. Six different cases are discussed

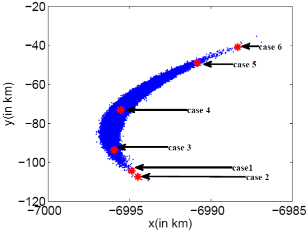
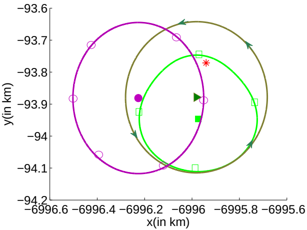
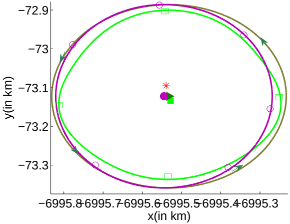


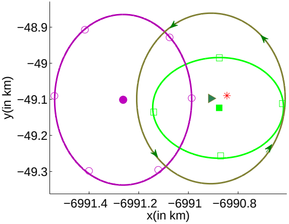
Figure: *Different measurement cases. Case 4 corresponds to prior mean*



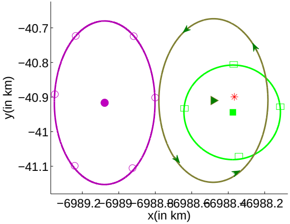
(a) case 3



(b) case 4(measurements = prior mean)



(c) case 5



(d) case 6

Filters	Pos. Error In Position (km)	Pos. Variance of Position (km^2)
AGSF	0.237	0.7837
UKF	0.244	0.0339
EKF	1.3329	0.03194

Table: Norm of error/variance when only position measurements are available and they are not at prior mean (Pos. Error = Posterior Error, Pos. Variance = Posterior Variance)[2]

The EKF provides completely inconsistent estimates of orbital states.

[2] **K. Vishwajeet**, P. Singla and M. Jah, "Nonlinear Uncertainty Propagation for Perturbed Two-Body Orbits," *AIAA Journal of Guidance, Control and Dynamics*, January 2014, DOI: 10.2514/1.G000472.

Scenario II: Range, azimuthal and elevation angles are available for measurement

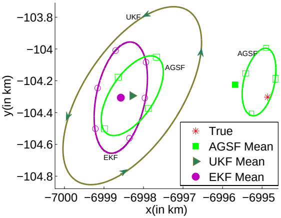
Measurement model is:

$$\mathbf{z}_{k+1}(1) = \sqrt{(\mathbf{x}_{k+1}(1)-\mathbf{rSite}(1))^2+(\mathbf{x}_{k+1}(2)-\mathbf{rSite}(2))^2+(\mathbf{x}_{k+1}(3)-\mathbf{rSite}(3))^2}+\boldsymbol{\nu}_1$$

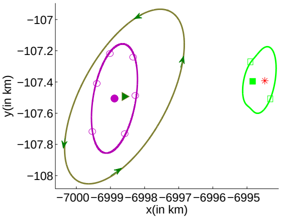
$$\mathbf{z}_{k+1}(2) = \tan^{-1}\frac{\mathbf{x}_{k+1}(3)-\mathbf{rSite}(3)}{\sqrt{(\mathbf{x}_{k+1}(1)-\mathbf{rSite}(1))^2+(\mathbf{x}_{k+1}(2)-\mathbf{rSite}(2))^2}}+\boldsymbol{\nu}_2$$

$$\mathbf{z}_{k+1}(3) = \tan^{-1}\frac{\mathbf{x}_{k+1}(2)-\mathbf{rSite}(2)}{\mathbf{x}_{k+1}(1)-\mathbf{rSite}(1)}+\boldsymbol{\nu}_3$$

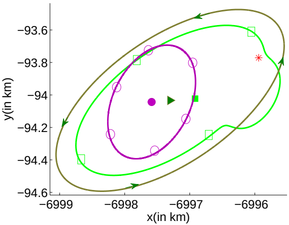
$$\textit{where,}\quad \boldsymbol{\nu} = \begin{bmatrix}\boldsymbol{\nu}_1 & \boldsymbol{\nu}_2 & \boldsymbol{\nu}_3\end{bmatrix}^T \sim \mathcal{N}(\mathbf{0},\mathbf{R}_2) \quad \text{and,} \quad \mathbf{R}_2 = diag\left(\underbrace{0.01}_{km^2} \quad \underbrace{0.0174 \quad 0.0174}_{rad^2}\right)$$



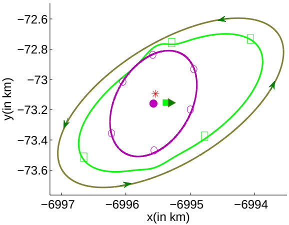
(a) case 1



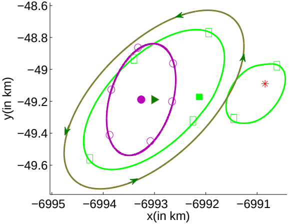
(b) case 2



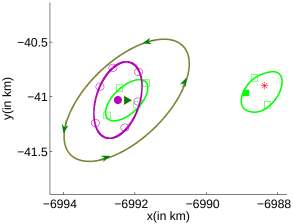
(a) case 3



(b) case 4(measurements = prior mean)



(c) case 5



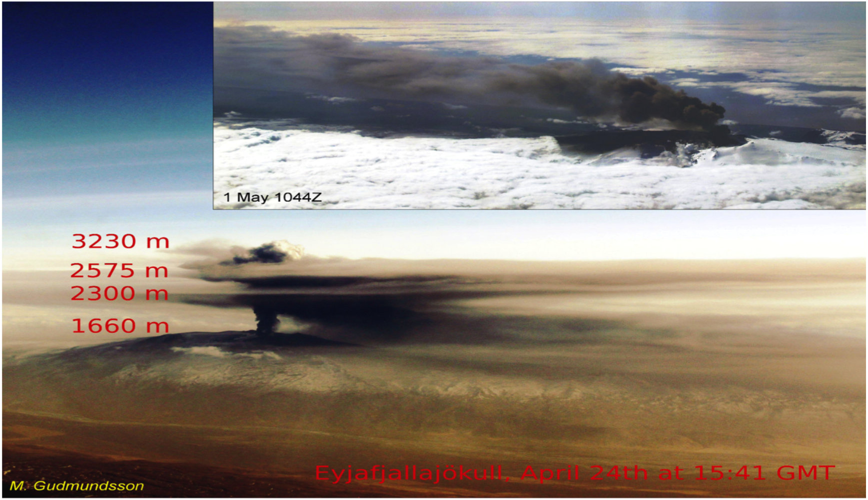
(d) case 6

Filters	Pos. Error In Position (km)	Pos. Variance of Position (km^2)
AGSF	2.042	3.4608
UKF	7.1679	3.1315
EKF	7.7479	0.3987

Table: Norm of error when range and angular measurements are available and they are not at prior mean (Pos. Error = Posterior Error, Pos. Variance = Posterior Variance)[3]

The EKF provides completely inconsistent estimates of orbital states.

[3] **K. Vishwajeet**, P. Singla and M. Jah, "Nonlinear Uncertainty Propagation for Perturbed Two-Body Orbits," *AIAA Journal of Guidance, Control and Dynamics*, January 2014, DOI: 10.2514/1.G000472.


1 May 1044Z
3230 m
2575 m
2300 m
1660 m
M. Gudmundsson
Eyjafjallajökull, April 24th at 15:41 GMT

- The `BENT` integral eruption column model was used to produce eruption column parameters (mass loading, column height, grain size distribution) given *a specific atmospheric sounding and source conditions.*
  - `BENT` takes into consideration atmospheric (wind) conditions as given by atmospheric sounding data.
  - Plume rise height is given as *a function of volcanic source and environmental conditions.*
- The `PUFF` Lagrangian model was used to propagate ash parcels in *a given wind field (NCEP Reanalysis).*
  - `PUFF` takes into account dry deposition as well as dispersion and advection.
- Polynomial chaos quadrature (PCQ) was used to select sample points and weights in the uncertain input space of vent radius, vent velocity, mean particle size and particle size variance.

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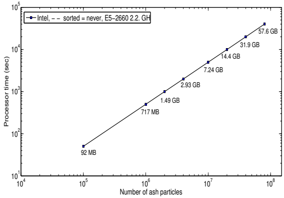
TABLE: Eruption source parameters based on observations of Eyjafjallajökull volcano and information from other similar eruptions.

Parameter	Value range	PDF	Comment
Vent radius, b_0, m	65-150	Uniform	Measured from radar image of summit vents
Vent velocity, w_0, m/s	Range: 45-124	Uniform	M. Ripepe, Geneva, Switzerland, 2010, presentation
Mean grain size, Md_φ	3.5-7	Uniform	Woods and Bursik (1991), Table 1, vulcanian and phreatoplinian. A. Hoskuldsson, Iceland meeting 2010, presentation
σ_φ	$0.5 - 3$	Uniform	Woods and Bursik (1991), Table 1, vulcanian and phreatoplinian.

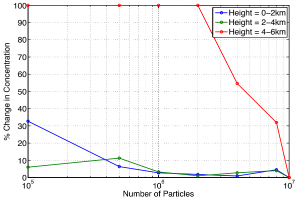
- 4×10^7 particles were used in BENT-PUFF model.

Location 52N, 13.5E: conc is the puff computed absolute air concentration (in mg/m^3) in a grid cell of size $0.5^\circ \times 0.5^\circ \times 2km$ at 1200hours on 16^{th} April, 2010 , and count is the number of PUFF particles in that cell

# of particles	height (km)	conc.$\times 10^{-4}$	count	height	conc.$\times 10^{-5}$	count	height	conc.$\times 10^{-7}$	count
10^5	3	0.74	28	5	4.23	16	7	-	-
5×10^5	3	1.17	221	5	3.54	67	7	-	-
10^6	3	1.12	405	5	4.12	156	7	-	-
2×10^6	3	1.12	884	5	4.03	305	7	-	-
4×10^6	3	1.09	1655	5	4.10	3620	7	1.32	2
8×10^6	3	1.15	3471	5	4.15	1256	7	1.98	6
10^7	3	1.10	4151	5	3.99	1510	7	2.91	11



Processor Time for Single Deterministic Run of BENT-PUFF vs. Number of Ash Particles.



Concentration (52N 13.5E) vs. Number of PUFF Particles

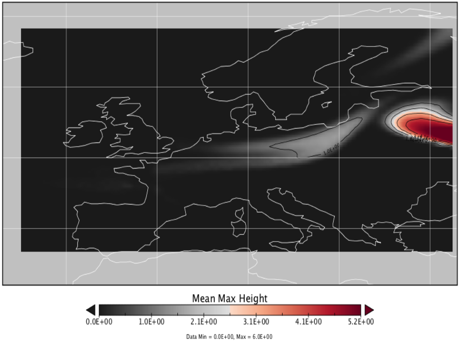


Figure: 9^4 *Clenshaw Curtis Runs*

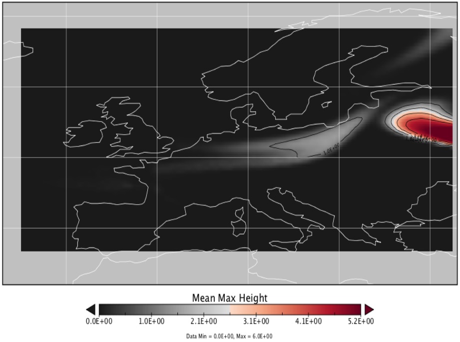


Figure: 161 *CUT Runs*

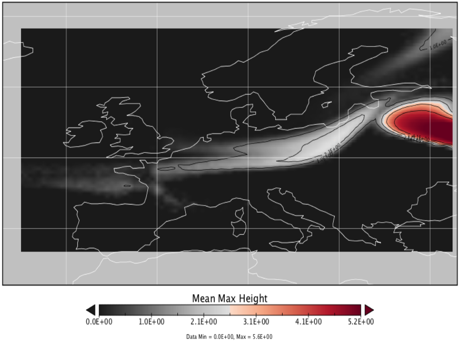


Figure: 9^4 *Clenshaw Curtis Runs*

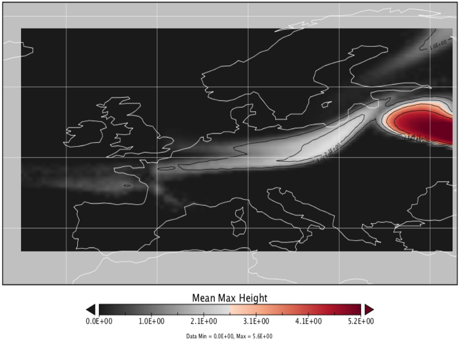
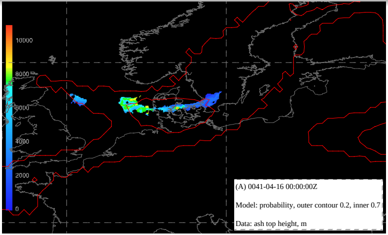


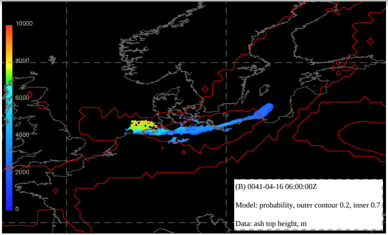
Figure: 161 *CUT Runs*

Probability of Ash Top Height

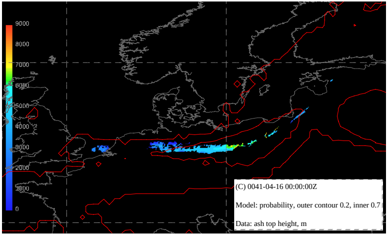
Iceland Volcano (Eyjafjallajökull) Eruption



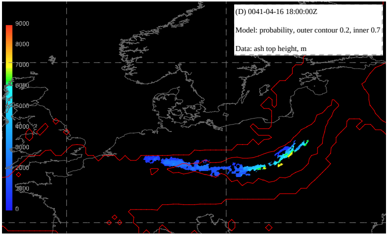
(a) 00 hrs



(b) 06 hrs



(c) 12 hrs



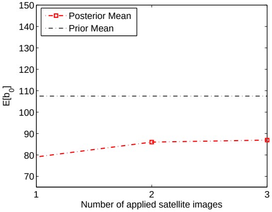
(d) 18 hrs

R. Madankan, et al., "Computation of Probabilistic Hazard Maps and Source Parameter Estimation For Volcanic Ash Transport and Diffusion," *Journal of Computational Physics*, DOI: 10.1016/j.jcp.2013.11.032

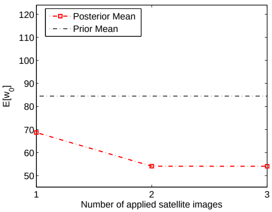
Source Parameter Estimation

- Ash top-height (obtained from satellite imagery) is used as measurement data.
- Satellite data from three different time instants (April 16^{th} at 0600 hrs, 1200 hrs, and 1800 hrs) are used as measurement data.
- Satellite observed ash top-heights are *assumed* to be accurate to within 100 m intervals around the observed height.
- Due to height quantization in the bent-puff model, ash top-height provided by bent-puff model is assumed to be polluted with zero-mean uniformly distributed random noise between $-1000\ m$ and $+1000\ m$.
- Minimum variance framework was used for source parameter estimation.

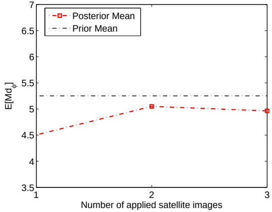
Source Parameter Estimation



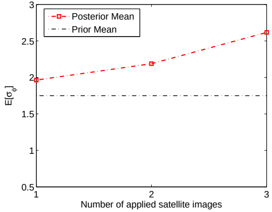
(e) Vent Radius



(f) Vent Velocity



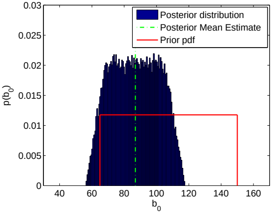
(g) Mean Grain Size



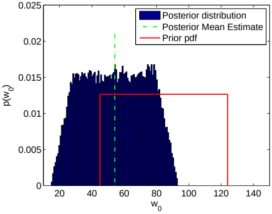
(h) σ_ϕ

Application to Ash Dispersion Problem

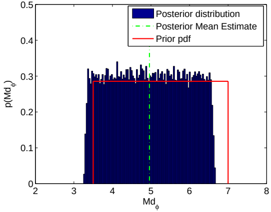
Source Parameter Estimation



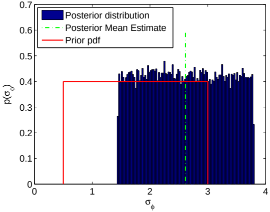
(i) Vent Radius



(j) Vent Velocity

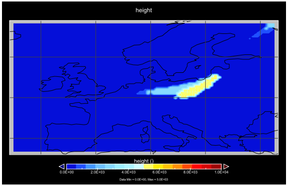


(k) Mean Grain Size

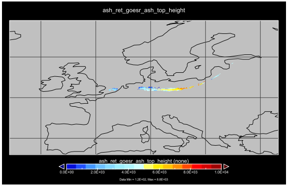


(l) σ_ϕ

Comparison of Forecast of Ash top-height and Satellite Observation on ***April*** **16th, 1200 hrs.**

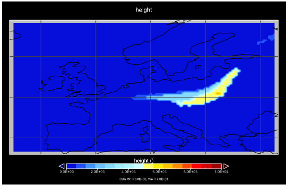


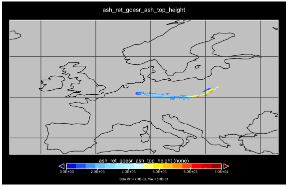
(a) Model Forecast



(b) Satellite Observation

Comparison of Forecast of Ash top-height and Satellite Observation on ***April*** **16th, 1800 hrs.**





- **Non-Gaussian Approximation:**
  - AGSF > Quadrature & CUT based filters > UKF > EKF
- **Computational Cost:**
  - AGSF > Quadrature & CUT based filters > UKF $\sim$ EKF
- **Ease of Implementation:**
  - Quadrature & CUT based filters $\sim$ UKF > EKF > AGSF

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  - Quadrature & CUT based filters $\sim$ UKF > EKF > AGSF

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Acknowledgement

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