Propagation of PDF by solving the Kolmogorov Equation

Puneet Singla, Manoranjan Majji
Department of Aerospace Engineering
Pennsylvania State University, University Park, PA-16802
Texas A&M University, College Station, TX-77843

Workshop: New Advances in Uncertainty Analysis & Estimation

Air-force Research Laboratories, Kirtland, NM

July 18-19, 2017

Acknowledgement: K. Vishwajeet & N. Adurthi

KOLMOGOROV EQUATION

- *Discrete System:* $\mathbf{x}_{k+1} = \Phi(\mathbf{x}_k, t_k, t_{k+1}) + \mathbf{w}_k$

$$p(\mathbf{x}_{k+1}) = \int p(\mathbf{x}_{k+1} \mid \mathbf{x}_k) p(\mathbf{x}_k) d\mathbf{x}_k$$

$$p(\mathbf{x}_{k+1} \mid \mathbf{x}_k) = p_{\mathbf{w}_k}(\mathbf{x}_{k+1} - \Phi(\mathbf{x}_k, t_k, t_{k+1}))$$

- *Continuous System:* $d\mathbf{x}_t = f(\mathbf{x}_t, t)dt + G(\mathbf{x}_t, t)d\beta_t$

$$\frac{\partial p(x,t)}{\partial t} = \underbrace{-\sum_{i=1}^{n} \frac{\partial p f_i}{\partial x_i}}_{\text{Drift Term}} + \underbrace{\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\left(\partial^2 p [GQG]_{ij}\right)}{\partial x_i \partial x_j}}_{\text{Diffusion Term}}$$

This is the differential form of the CKE which is known as the Fokker-Planck-Kolmogorov Equation (FPKE) or *Kolmogorov's Forward Equation*.

Linear System: $\mathbf{x}_{k+1} = F_k \mathbf{x}_k + v_k$

Assumptions: $\mathbf{x}_0 \sim \mathcal{N}(\mathbf{x}_0 : \hat{\mathbf{x}}_0, P_0)$, $v_k \sim \mathcal{N}(v_k : \mathbf{0}, Q_k)$

Time Evolution: $p(\mathbf{x}_{k+1}) = \int \mathcal{N}(\mathbf{x}_{k+1} : F_k \mathbf{x}_k, Q_k) \,.\, \mathcal{N}(\mathbf{x}_k : \hat{\mathbf{x}}_k, P_k)\, d\mathbf{x}_k$

$$
\begin{aligned}
p(\mathbf{x}_{k+1}) &= \int \frac{1}{\sqrt{|2\pi Q_k|}} \exp\left[-\frac{1}{2}\left(\mathbf{x}_{k+1} - F_k \mathbf{x}_k\right)^T Q_k^{-1} \left(\mathbf{x}_{k+1} - F_k \mathbf{x}_k\right)\right] \\
&\qquad \cdot \frac{1}{\sqrt{|2\pi P_k|}} \exp\left[-\frac{1}{2}\left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)^T P_k^{-1} \left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)\right] d\mathbf{x}_k \\
&= \int \frac{1}{\sqrt{|2\pi Q_k|}} \cdot \frac{1}{\sqrt{|2\pi P_k|}} \cdot \exp\left[-\frac{1}{2}\left(\mathbf{x}_{k+1} - F_k \mathbf{x}_k\right)^T Q_k^{-1} \left(\mathbf{x}_{k+1} - F_k \mathbf{x}_k\right)\right. \\
&\qquad \left. - \frac{1}{2}\left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)^T P_k^{-1} \left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)\right] d\mathbf{x}_k
\end{aligned}
$$

Simplifying the exponent

$$
-\frac{1}{2}\left(\mathbf{x}_{k+1} - F_k \mathbf{x}_k\right)^T Q_k^{-1} \left(\mathbf{x}_{k+1} - F_k \mathbf{x}_k\right) - \frac{1}{2}\left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)^T P_k^{-1} \left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)
$$

Simplifying the exponent

$$
\begin{aligned}
&-\frac{1}{2}\left(\mathbf{x}_{k+1}-F_k\mathbf{x}_k\right)^T Q_k^{-1}\left(\mathbf{x}_{k+1}-F_k\mathbf{x}_k\right)-\frac{1}{2}\left(\mathbf{x}_k-\hat{\mathbf{x}}_k\right)^T P_k^{-1}\left(\mathbf{x}_k-\hat{\mathbf{x}}_k\right)\\
&=-\frac{1}{2}\left(\mathbf{x}_k^T\underbrace{\left[P_k^{-1}+F_k^T Q_k^{-1}F_k\right]}_{A}\mathbf{x}_k-2\underbrace{\left[\hat{\mathbf{x}}_k^T P_k^{-1}+\mathbf{x}_{k+1}^T Q_k^{-1}F_k\right]}_{c^T}\mathbf{x}_k+\mathbf{x}_{k+1}^T Q_k^{-1}\mathbf{x}_{k+1}+\hat{\mathbf{x}}_k^T P_k^{-1}\hat{\mathbf{x}}_k\right)
\end{aligned}
$$

$$
\begin{aligned}
p(\mathbf{x}_{k+1})=\frac{1}{\sqrt{|2\pi Q_k|}}\cdot\frac{1}{\sqrt{|2\pi P_k|}}\cdot\exp\left[-\frac{1}{2}\mathbf{x}_{k+1}^T Q_k^{-1}\mathbf{x}_{k+1}-\frac{1}{2}\hat{\mathbf{x}}_k^T P_k^{-1}\hat{\mathbf{x}}_k\right]\\
\int\exp\left[-\frac{1}{2}\mathbf{x}_k^T A\mathbf{x}_k+c^T\mathbf{x}_k\right]d\mathbf{x}_k
\end{aligned}
$$

Using the identity $\int\exp\left[-\frac{1}{2}x^T Ax+c^T x\right]dx=\sqrt{|2\pi A^{-1}|}\exp\left[\frac{1}{2}c^T A^{-T}c\right]$

$$
p(\mathbf{x}_{k+1})=\frac{1}{\sqrt{|2\pi Q_k|}}\cdot\frac{\sqrt{|2\pi A^{-1}|}}{\sqrt{|2\pi P_k|}}\cdot\exp\left[-\frac{1}{2}\mathbf{x}_{k+1}^T Q_k^{-1}\mathbf{x}_{k+1}-\frac{1}{2}\hat{\mathbf{x}}_k^T P_k^{-1}\hat{\mathbf{x}}_k+\frac{1}{2}c^T A^{-T}c\right]
$$

Simplifying the exponent further gives

$$\Rightarrow -\frac{1}{2}\mathbf{x}_{k+1}^T Q_k^{-1}\mathbf{x}_{k+1} - \frac{1}{2}\hat{\mathbf{x}}_k^T P_k^{-1}\hat{\mathbf{x}}_k + \frac{1}{2}c^T A^{-T} c$$

$$\Rightarrow -\frac{1}{2}\mathbf{x}_{k+1}^T Q_k^{-1}\mathbf{x}_{k+1} - \frac{1}{2}\hat{\mathbf{x}}_k^T P_k^{-1}\hat{\mathbf{x}}_k + \frac{1}{2}\left[\hat{\mathbf{x}}_k^T P_k^{-1} + \mathbf{x}_{k+1}^T Q_k^{-1} F_k\right]\left[P_k^{-1} + F_k^T Q_k^{-1} F_k\right]^{-T}\left[\hat{\mathbf{x}}_k^T P_k^{-1} + \mathbf{x}_{k+1}^T Q_k^{-1} F_k\right]^T$$

$$\Rightarrow -\frac{1}{2}\mathbf{x}_{k+1}^T \underbrace{\left[Q_k^{-1} - Q_k^{-1}F_k\left(P_k^{-1} + F_k^T Q_k^{-1} F_k\right)^{-1} F_k^T Q_k^{-1}\right]}_{C}\mathbf{x}_{k+1} + \underbrace{\hat{\mathbf{x}}_k^T P_k^{-1}\left(P_k^{-1} + F_k^T Q_k^{-1} F_k\right)^{-1} F_k^T Q_k^{-1}}_{b^T}\mathbf{x}_{k+1}$$

$$-\frac{1}{2}\hat{\mathbf{x}}_k^T \underbrace{\left[P_k^{-1} - P_k^{-1}\left(P_k^{-1} + F_k^T Q_k^{-1} F_k\right)^{-1} P_k^{-1}\right]}_{D}\hat{\mathbf{x}}_k$$

$$\Rightarrow -\frac{1}{2}\mathbf{x}_{k+1}^T\, C\, \mathbf{x}_{k+1} + b^T\mathbf{x}_{k+1} - \frac{1}{2}\hat{\mathbf{x}}_k^T\, D\, \hat{\mathbf{x}}_k$$

$$\Rightarrow -\frac{1}{2}\left(\mathbf{x}_{k+1} - C^{-1}b\right)^T C\left(\mathbf{x}_{k+1} - C^{-1}b\right) + \frac{1}{2}b^T C^{-1} b - \frac{1}{2}\hat{\mathbf{x}}_k^T\, D\, \hat{\mathbf{x}}_k$$

$$p(\mathbf{x}_{k+1}) = \frac{1}{\sqrt{(2\pi)^n |Q_k||P_k||A|}} \cdot \exp\left[-\frac{1}{2}\left(\mathbf{x}_{k+1} - C^{-1}b\right)^T C \left(\mathbf{x}_{k+1} - C^{-1}b\right) + \frac{1}{2} b^T C^{-1} b - \frac{1}{2}\hat{\mathbf{x}}_k^T D \hat{\mathbf{x}}_k\right]$$

$$C = \left[Q_k^{-1} - Q_k^{-1} F_k \left(P_k^{-1} + F_k^T Q_k^{-1} F_k\right)^{-1} F_k^T Q_k^{-1}\right] \qquad b^T = \hat{\mathbf{x}}_k^T P_k^{-1} \left(P_k^{-1} + F_k^T Q_k^{-1} F_k\right)^{-1} F_k^T Q_k^{-1}$$

$$C^{-1} = Q_k + F_k P_k F_k^T \qquad C^{-1} b = F_k \hat{\mathbf{x}}_k$$

$$b^T C^{-1} b = \hat{\mathbf{x}}_k^T P_k^{-1} \left(P_k^{-1} + F_k^T Q_k^{-1} F_k\right)^{-1} F_k^T Q_k^{-1} F \hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^T F_k^T \left(F_k P_k F_k^T + Q_k\right)^{-1} F_k \hat{\mathbf{x}}_k$$

$$D = P_k^{-1} - P_k^{-1} \left(P_k^{-1} + F_k^T Q_k^{-1} F_k\right)^{-1} P_k^{-1} = F_k^T \left(F_k P_k F_k^T + Q_k\right)^{-1} F_k$$

$$|Q_k||P_k||A| = |Q_k||P_k|\left|\left(P_k^{-1} + F_k^T Q_k^{-1} F_k\right)\right| = |F_k P_k F^T + Q_k| \quad \textit{(by Sylvester's determinant theorem)}$$

$$p(\mathbf{x}_{k+1}) = \frac{1}{\sqrt{(2\pi)^n |F_k P_k F_k^T + Q_k|}} \cdot \exp\left[-\frac{1}{2}\left(\mathbf{x}_{k+1} - F_k \hat{\mathbf{x}}_k\right)^T \left(F_k P_k F_k^T + Q_k\right)^{-1} \left(\mathbf{x}_{k+1} - F_k \hat{\mathbf{x}}_k\right)\right]$$

The prior pdf $p(\mathbf{x}_{k+1})$ has mean $\hat{\mathbf{x}}_{k+1}$ and covariance P_{k+1} given as

$$\hat{\mathbf{x}}_{k+1} = F_k \hat{\mathbf{x}}_k$$

$$P_{k+1} = F_k P_k F_k^T + Q_k$$

Time Evolution

$$p(\mathbf{x}_{k+1}) = \frac{1}{\sqrt{(2\pi)^n |P_{k+1}|}} \cdot \exp\left[-\frac{1}{2} \left(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}\right)^T P_{k+1}^{-1} \left(\mathbf{x}_{k+1} \hat{\mathbf{x}}_{k+1}\right) \right]$$

The pdf $p(\mathbf{x}_{k+1})$ has mean $\hat{\mathbf{x}}_{k+1}$ and covariance P_{k+1} given as

$$\hat{\mathbf{x}}_{k+1} = F_k \hat{\mathbf{x}}_k$$
$$P_{k+1} = F_k P_k F^T + Q_k$$

Hence, the prior pdf remains Gaussian

As we know the pdf remains Gaussian, an easier approach is to directly compute the mean and covariance from the linear model equations as:

$$\hat{\mathbf{x}}_{k+1} = E[\mathbf{x}_{k+1}] = E\left[F_k \mathbf{x}_k\right] + E[\nu_k] = F_k E\left[\mathbf{x}_k\right] = F_k \hat{\mathbf{x}}_k$$
$$P_{k+1} = E\left[\left(\mathbf{x}_{k+1} - F_k \hat{\mathbf{x}}_k\right)\left(\mathbf{x}_{k+1} - F_k \hat{\mathbf{x}}_k\right)^T\right] = E\left[\left(F_k \mathbf{x}_k + \nu_k - F_k \hat{\mathbf{x}}_k\right)\left(F_k \mathbf{x}_k + \nu_k - F_k \hat{\mathbf{x}}_k\right)^T\right]$$
$$= E\left[F_k (\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T F_k^T + \nu_k \nu_k^T\right] = F_k P_k F^T + Q_k$$

Time Evolution

$$p(\mathbf{x}(t)) = \frac{1}{\sqrt{(2\pi)^n |P(t)|}} \cdot \exp\left[-\frac{1}{2}\left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)^T P(t)^{-1} \left(\mathbf{x}(t) - \hat{\mathbf{x}}(t)\right)\right]$$

The pdf $p(\mathbf{x}(t)$ has mean $\hat{\mathbf{x}}(t)$ and covariance $P(t)$ given as:

$$\dot{\hat{x}}(t) = F(t)\hat{\mathbf{x}}(t)$$
$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + Q(t)$$

Hence, the prior pdf remains Gaussian

Consider the following $n-$ dimensional dynamical system

$$d\mathbf{x}(t) = f(\mathbf{x}(t),t)dt + G(\mathbf{x}(t),t)d\beta(t)$$
$$\mathrm{E}[d\beta(t)\beta^T(t)] = \mathbf{Q}(t)dt \qquad p(\mathbf{x}(t_0)) = \mathscr{N}(\mathbf{x}|\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$$

Problem statement: Find pdf, $p(\mathbf{x},t)\ \forall \mathbf{x}, t > t_0$.

Rate of change of $p(\mathbf{x},t)$ is given by Fokker Planck Kolmogorov Equation (FPKE):

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = \mathscr{L}_{\mathscr{FP}}(p(\mathbf{x},t)) = \sum_{i=1}^{n} \frac{\partial (pf_i)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\left[\left(\mathbf{GQG}^T\right)_{ij} p\right]}{\partial x_i \partial x_j}$$

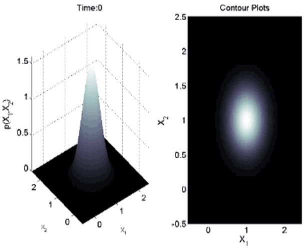


FIGURE: evolution of pdf

Inherent Difficulties in numerically solving the FPKE

- Positivity and normality constraint of pd:f
  - $p(\mathbf{x},t) \geq 0, \forall\, \mathbf{x}, t$ and, $\int_{R^n} p(\mathbf{x},t) d\mathbf{X} = 1$
- No unique domain for enforcing boundary conditions:
  - Domain of FPKE solution is $(-\infty, \infty)^n$.
  - Numerical methods require finite domain.
- Number of spatial coordinates.
  - FPKE contains the partial derivatives w.r.t. all the states.
  - For a $2-$ dimensional rigid body motion, we need discretization in 6 dimensions.

Solution of FPKE for stationary pdf has been derived for a restricted class of dynamical systems: A.T.Fuller[1], Stratonovich[2]

[1] Analysis of nonlinear stochastic systems by means of Fokker-Planck equation

[2] Topics in the theory of random noise

Inherent Difficulties in numerically solving the FPKE

- Positivity and normality constraint of pd:f
  - $p(\mathbf{x},t) \geq 0, \forall\ \mathbf{x}, t$ and, $\int_{\mathrm{R}^n} p(\mathbf{x},t)d\mathbf{X} = 1$
- No unique domain for enforcing boundary conditions:
  - Domain of FPKE solution is $(-\infty,\infty)^n$.
  - Numerical methods require finite domain.
- Number of spatial coordinates.
  - FPKE contains the partial derivatives w.r.t. all the states.
  - For a $2-$ dimensional rigid body motion, we need discretization in 6 dimensions.

Solution of FPKE for stationary pdf has been derived for a restricted class of dynamical systems: A.T.Fuller[1], Stratonovich[2]

How to find FPKE solution for non-stationary pdf

[1] Analysis of nonlinear stochastic systems by means of Fokker-Planck equation

[2] Topics in the theory of random noise

Solution of FPKE
Stationary PDF

Let there be an $n-$ dimensional system described by $\mathbf{F} = \mathbf{ma}$.

If F is conservative, then there exists a potential $\mathbf{V}$, such that

$$F = -\nabla \mathbf{V}$$

$$\frac{dx_i}{dt} = v_i; \qquad m\frac{dv_i}{dt} = -\nabla_i \mathbf{V}$$

Let us define new coordinates as follows:

Configuration Variables→ $q_i = x_i$

Conjugate Momenta→ $p_i = m_i \dot{x}_i = m_i v_i$

If we define $H(q,p)$ as followed,

$H(q,p)$ = Hamiltonian

$$H(q,p) = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(q)$$

then, following equations hold

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Let there be an $n-$ dimensional system described by $\mathbf{F} = \mathbf{ma}$.

If F is conservative, then there exists a potential $\mathbf{V}$, such that

$$F = -\nabla \mathbf{V}$$

$$\frac{dx_i}{dt} = v_i; \qquad m\frac{dv_i}{dt} = -\nabla_i \mathbf{V}$$

Let us define new coordinates as follows:

Configuration Variables$\rightarrow$ $q_i = x_i$

Conjugate Momenta$\rightarrow$ $p_i = m_i \dot{x}_i = m_i v_i$

If we define $H(q,p)$ as followed,

$H(q,p)$ = Hamiltonian

$$H(q,p) = \sum_{i=1}^{n} \frac{p_i^2}{2m_i} + V(q)$$

then, following equations hold

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Now, let us calculate $\frac{dH}{dt}$.

$$\begin{aligned}\frac{dH}{dt} &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} \\ &= \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} \\ &= 0\end{aligned}$$

$H(q,p)$ is conserved.

For an $n-$ dimensional dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $n = 2N$, where, first N components correspond to position$(\mathbf{x})$ and the last N to velocity$(\dot{\mathbf{x}})$.

Let $y_i = \dot{\mathbf{x}}_i$. i.e.

$$\frac{dx_i}{dt} = f_i; \qquad i = 1, \cdots, N$$
$$\frac{dy_i}{dt} = f_{i+N}; \qquad i = 1, \cdots, N$$

Using the Hamiltonian,

$$\frac{dx_i}{dt} = \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$
$$\frac{dy_i}{dt} = \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Using FPKE and substituting for f_i using Hamiltonian,

$$\begin{aligned}
\frac{\partial p(\mathbf{x},t)}{\partial t} &= -\sum_{i=1}^{n} \frac{\partial (p f_i)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \left[\left(\mathbf{g}\mathbf{Q}\mathbf{g}^T\right)_{ij} p \right]}{\partial x_i \partial x_j} \\
&= -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(p\frac{dx_i}{dt}\right) + \frac{\partial}{\partial y_i}\left(p\frac{dy_i}{dt}\right) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \left[\left(\mathbf{g}\mathbf{Q}\mathbf{g}^T\right)_{ij} p \right]}{\partial x_i \partial x_j} \\
&= -\sum_{i=1}^{N} \left(\frac{\partial}{\partial q_i}\left(p\frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p_i}\left(p\frac{\partial H}{\partial q_i}\right)\right) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 \left[\left(\mathbf{g}\mathbf{Q}\mathbf{g}^T\right)_{ij} p \right]}{\partial x_i \partial x_j}
\end{aligned}$$

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{N}\left(\frac{\partial}{\partial q_i}\left(p\frac{\partial H}{\partial p_i}\right) - \frac{\partial}{\partial p_i}\left(p\frac{\partial H}{\partial q_i}\right)\right) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\left[\left(\mathbf{gQg}^T\right)_{ij}p\right]}{\partial x_i\partial x_j}$$

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\sum_{i=1}^{N}\left(\frac{\partial p}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p}{\partial p_i}\frac{\partial H}{\partial q_i} + p\frac{\partial^2 H}{\partial p_i\partial q_i} - p\frac{\partial^2 H}{\partial q_i\partial p_i}\right) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\left[\left(\mathbf{gQg}^T\right)_{ij}p\right]}{\partial x_i\partial x_j}$$

Let $p(\mathbf{x},t) = p(H(x_1,\cdots,x_N,y_1,\cdots,y_N)) = p(H(q_1,\cdots,q_N,p_1,\cdots,p_N))$.
Using the fact that for a stationary pdf $\frac{\partial p(\mathbf{x},t)}{\partial t} = 0$.

$$0 = -\sum_{i=1}^{N}\left(\frac{\partial p}{\partial H}\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p}{\partial H}\frac{\partial p}{\partial p_i}\frac{\partial H}{\partial q_i}\right) + \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\left[\left(\mathbf{gQg}^T\right)_{ij}p\right]}{\partial x_i\partial x_j}$$

If $\mathbf{Q} = 0$, any function of $H(q_1,\cdots,q_N,p_1,\cdots,p_N)$ will serve for $p(H(q_1,\cdots,q_N,p_1,\cdots,p_N))$ provided that it satisfies normality and boundary condition.

Otherwise $p(H(q_1,\cdots,q_N,p_1,\cdots,p_N))$ is found by solving

$$\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2\left[\left(\mathbf{gQg}^T\right)_{ij}p\right]}{\partial x_i\partial x_j} = 0$$

Let us consider the following example of a duffing oscillator.

$$\ddot{x} + \eta\dot{x} + \alpha x + \beta x^3 = Q;$$

where, $\alpha = -1; \beta = 3; \eta = 10$. In state state form, it can be written as:

$$\dot{x}_1 = x_2;$$
$$\dot{x}_2 = \mathbf{Q} - \eta x_2 - \alpha x_1 - \beta x_1^3$$

We define the following Hamiltonian kind function: Similar to integrating it.

$$H = \frac{x_2^2}{2} + \frac{\alpha x_1^2}{2} + \frac{\beta x_1^4}{4}$$

Remember P.E. $(V) = -\int F.dx$
and, $H =$ K.E. $+$ P.E.

Then, the equation of motion can be written as:

$$\frac{dx_1}{dt} = \frac{\partial H}{\partial x_2};$$
$$\frac{dx_2}{dt} = -\frac{\partial H}{\partial x_1} - \eta\frac{\partial H}{\partial x_2} + Q$$

$$\frac{dx_1}{dt} = \frac{\partial H}{\partial x_2}; \qquad \frac{dx_2}{dt} = -\frac{\partial H}{\partial x_1} - \eta \frac{\partial H}{\partial x_2} + Q$$

Writing FPKE for this system, we get,

$$\underbrace{-\frac{\partial}{\partial x_1}\left(p\frac{\partial H}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(p\frac{\partial H}{\partial x_1}\right)}_{=0 \text{ (As calculated earlier)}} + \eta \frac{\partial}{\partial x_2}\left(p\frac{\partial H}{\partial x_2}\right) + \frac{1}{2}Q\frac{\partial^2 p}{\partial x_2^2} = 0$$

Further calculations:

$$\eta \frac{\partial}{\partial x_2}\left(p\frac{\partial H}{\partial x_2}\right) + \frac{1}{2}Q\frac{\partial^2 p}{\partial x_2^2} = 0$$

$$\eta p\frac{\partial H}{\partial x_2} + \frac{1}{2}Q\frac{\partial p}{\partial x_2} = C \quad \text{(constant)}$$

Using $p(\pm\infty, t) = 0$,

$$\eta p\frac{\partial H}{\partial x_2} + \frac{1}{2}Q\frac{\partial p}{\partial x_2} = 0$$

Further simplifications:

$$\eta p\frac{\partial H}{\partial x_2} + \frac{1}{2}Q\frac{\partial p}{\partial H}\frac{\partial H}{\partial x_2} = 0$$

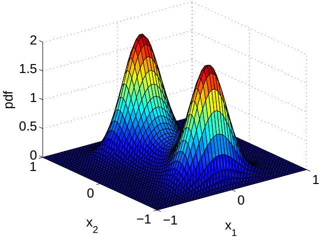
$$\eta p + \frac{1}{2}Q\frac{\partial p}{\partial H} = 0$$

$$p = p_0 \exp\left(-\frac{2\eta H}{Q}\right)$$

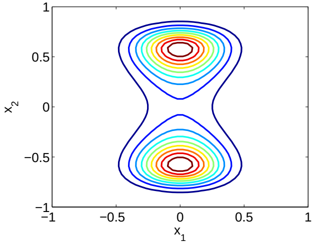
$$= p_0 \exp\left(-\frac{2\eta}{Q}\left(\frac{x_2^2}{2} + \frac{\alpha x_1^2}{2} + \frac{\beta x_1^4}{4}\right)\right)$$

$$p = p_0 \exp\left(-\frac{2\eta H}{Q}\right)$$
$$= p_0 \exp\left(-\frac{2\eta}{Q}\left(\frac{x_2^2}{2} + \frac{\alpha x_1^2}{2} + \frac{\beta x_1^4}{4}\right)\right)$$

Using $\int_{-\infty}^{\infty} p(t,\mathbf{x})\, d\mathbf{x} = 1$, we get, $p_0 = 2.6633$.



(a) pdf



(b) contour

FIGURE: pdf and contours using analytical expression

Finally, we consider the following example of a damping oscillator.

$$\ddot{x} + \beta\dot{x} + x + \alpha\left(x^2 + \dot{x}^2\right)\dot{x} = g\nu; \qquad \nu \sim \mathcal{N}(0, Q)$$

where, $\beta = -0.5; \alpha = 0.125$ and, $Q = \frac{1}{\pi}; g = 1$.
The stationary pdf can be written as:

$$p(x, \dot{x}) \propto \exp\left(-\frac{\eta}{2g^2}\left(\beta(x^2 + \dot{x}^2) + \frac{\alpha}{2}(x^2 + \dot{x}^2)^2\right)\right)$$

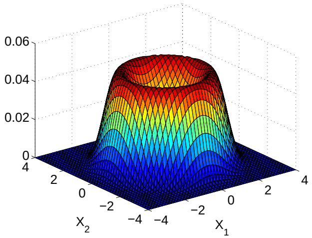


FIGURE: pdf using analytical expression

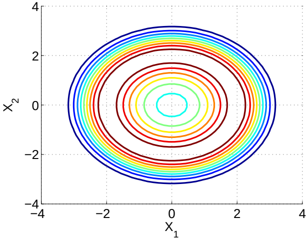
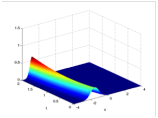


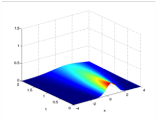
FIGURE: contours using analytical expression

With sufficient number of Gaussian components, any pdf can be approximated as closely as desired.



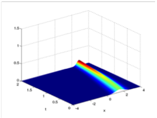
Component 1

+



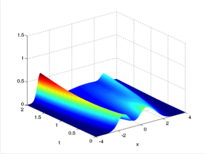
Component 2

+



Component 3

=



$$\hat{p}(\mathbf{x},t) \approx \sum_{i=1}^{\infty} w_t^i p_{g_i} \qquad p_{g_i} \sim \mathcal{N}\left(\mathbf{x}|\boldsymbol{\mu}_t^i, \boldsymbol{\Sigma}_t^i,\right) \qquad \sum_i w_t^i = 1 \qquad w_t^i \geq 0$$

- Gaussian kernel is an ideal choice to approximate solution of the FPKE.
  - $p_{g_i}(\pm\infty) = 0$
- Question is how to find unknown parameters of this Gaussian Sum Mixture?

- Each Gaussian kernel capture the local behavior of non-Gaussian density function.

$$\dot{\boldsymbol{\mu}}_t^i = f(\boldsymbol{\mu}_t^i, t)$$

$$\dot{\boldsymbol{\Sigma}}_t^i = \mathbf{F}\boldsymbol{\Sigma}_t^i + \boldsymbol{\Sigma}_t^i\mathbf{F}^T + \mathbf{g}\mathbf{Q}\mathbf{g}^T, \; \mathbf{F} = \frac{\partial f(\mathbf{x},t)}{\partial \mathbf{x}}\Big|_{\mathbf{x}=\boldsymbol{\mu}_t^i}$$

- Update the weights of Gaussian Sum Mixture such that FPKE error is minimized. [3] [4]

$$\text{Residual Error}(e) = \frac{\partial \hat{p}}{\partial t} - \left(-\sum_{i=1}^{n} \frac{\partial\left(\hat{p} f_i\right)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\left[\left(\mathbf{g}\mathbf{Q}\mathbf{g}^T\right)_{ij}\hat{p}\right]}{\partial x_i \partial x_j} \right)$$

$$\text{where, } \hat{p} = \sum_{i=1}^{N} w_t^i p_{g_i}$$

[3] Terejanu et. al., "Uncertainty propagation for nonlinear dynamic systems using Gaussian mixture models", JGCD, 2008

[4] Terejanu et. al., "Adaptive Gaussian Sum Filter for Nonlinear Bayesian Estimation", TAC, 2011

$$e = \frac{\partial \hat{p}}{\partial t} - \left(-\sum_{i=1}^{n} \frac{\partial (\hat{p} f_i)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \left[(\mathbf{g}\mathbf{Q}\mathbf{g}^T)_{ij} \hat{p} \right]}{\partial x_i \partial x_j} \right); \qquad \hat{p} = \sum_{i=1}^{N} w_t^i p_{g_i}$$

Terejanu et al.[5], for the first time, proposed a method to update these weights even during pure propagation.

Δt = time diff. between two successive weight updates

$$\frac{\partial \hat{p}}{\partial t} = \frac{\partial}{\partial t} \sum_{i=1}^{N} w_t^i p_{g_i} \qquad \text{where, } p_{g_i} = \mathcal{N}\left(\mathbf{x}(t) | \boldsymbol{\mu}_t^i, \boldsymbol{\Sigma}_t^i\right)$$

$$= \sum_{i=1}^{N} \left(\dot{w}_t^i p_{g_i} + w_t^i \frac{\partial p_{g_i}^T}{\partial \boldsymbol{\mu}_t^i} \dot{\boldsymbol{\mu}}_t^i + w_t^i \mathrm{Tr}\left[\frac{\partial p_{g_i}}{\partial \boldsymbol{\Sigma}_t^i} \dot{\boldsymbol{\Sigma}}_t^i \right] \right)$$

$$\text{where, } \dot{w}_t^i = \frac{1}{\Delta t} \left(w_{t+\Delta t}^i - w_t^i \right)$$

Need to Replace $\hat{p} = \sum_{i=1}^{N} w_t^i p_{g_i}$ in last two terms of e

$$e = \sum_{i=1}^{N} \frac{1}{\Delta t} \left(w_{t+\Delta t}^i - w_t^i \right) p_{g_i} + w_t^i \frac{\partial p_{g_i}^T}{\partial \boldsymbol{\mu}_t^i} \dot{\boldsymbol{\mu}}_t^i + w_t^i \mathrm{Tr}\left[\frac{\partial p_{g_i}}{\partial \boldsymbol{\Sigma}_t^i} \dot{\boldsymbol{\Sigma}}_t^i \right] - \left(-\sum_{i=1}^{n} \frac{\partial (\hat{p} f_i)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \left[(\mathbf{g}\mathbf{Q}\mathbf{g}^T)_{ij} \hat{p} \right]}{\partial x_i \partial x_j} \right)$$

[5] Terejanu et al, JGCD 2008, TAC 2011

Following Terejanu et. al.[6]

$$e = \sum_{i=1}^{N} \frac{w^i_{t+\Delta t}}{\Delta t} p_{g_i} + \underbrace{w^i_t \left(\frac{\partial p_{g_i}^T}{\partial \boldsymbol{\mu}^i_t} \dot{\boldsymbol{\mu}}^i_t + \mathrm{Tr}\left[\frac{\partial p_{g_i}}{\partial \boldsymbol{\Sigma}^i_t} \dot{\boldsymbol{\Sigma}}^i_t \right] - \frac{p_{g_i}}{\Delta t} \right)}_{\mathcal{M}_{\mathcal{DT}}}$$

$$- \left(-\sum_{i=1}^{n} \frac{\partial (\hat{p} f_i)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \left[\left(\mathbf{gQg}^T \right)_{ij} \hat{p} \right]}{\partial x_i \partial x_j} \right)$$

Using $\hat{p} = \sum_{i=1}^{N} w^i_t p_{g_i}$, in the last two terms, we get,

$$-\sum_{i=1}^{n} \frac{\partial (\hat{p} f_i)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \left[\left(\mathbf{gQg}^T \right)_{ij} \hat{p} \right]}{\partial x_i \partial x_j}$$

$$= -\sum_{l=1}^{N} w^l_t \sum_{i=1}^{n} \frac{\partial (p_{g_l} f_i)}{\partial x_i} + \frac{1}{2} \sum_{l=1}^{N} w^l_t \sum_{i,j=1}^{n} \frac{\partial^2 \left[\left(\mathbf{gQg}^T \right)_{ij} p_{g_l} \right]}{\partial x_i \partial x_j} \qquad (1)$$

$$-\sum_{l=1}^{N} w^l_t \mathbf{f}^T \frac{\partial (p_{g_l})}{\partial \mathbf{x}} - w^l_t p_{g_l} \mathrm{tr} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{1}{2} \sum_{l=1}^{N} w^l_t \mathrm{tr} \left[\sum_{i,j=1}^{n} \frac{\partial^2 \left[\left(\mathbf{gQg}^T \right)_{ij} p_{g_l} \right]}{\partial x_i \partial x_j} \right]$$

$= \mathbf{w}_t^T \mathcal{L}_{\mathcal{FP}}$ where,

$$\mathcal{L}_{\mathcal{FP}} = -\sum_{l=1}^{N} \mathbf{f}^T \frac{\partial p_{g_l}}{\partial \mathbf{x}} - p_{g_l} \mathrm{tr} \frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{1}{2} \sum_{l=1}^{N} \mathrm{tr} \left[\sum_{i,j=1}^{n} \frac{\partial^2 \left[\left(\mathbf{gQg}^T \right)_{ij} p_{g_l} \right]}{\partial x_i \partial x_j} \right]$$

Finally,

Residual Error(e) =

$$\mathbf{p_g}^T \frac{\mathbf{w}_{t+\Delta t}}{\Delta t} + \mathbf{w}_t^T \left(\mathcal{M}_{\mathcal{DT}} - \mathcal{L}_{\mathcal{FP}} \right)$$

[6] Terejanu et al, JGCD 2008, TAC 2011

Residual Error$(e) =$

$$\mathbf{p_g}^T \frac{\mathbf{w}_{t+\Delta t}}{\Delta t} + \mathbf{w}_t^T \left(\mathscr{M}_{\mathscr{D}\mathscr{T}} - \mathscr{L}_{\mathscr{F}\mathscr{P}}\right)$$

$$\min_{w^i_{t+\Delta t}} \quad \mathbf{J} = \frac{1}{2}\int_V e^2\left(t,\mathbf{x}\right) d\mathbf{x}$$

$$\text{subject to:} \quad \sum_{i=1}^{N} w^i_{t+\Delta t} = 1$$

$$w^i_{t+\Delta t} \geq 0, i = 1, 2, \ldots, N$$

$$\mathbf{J} = \frac{1}{2}\int_V \left(\mathbf{p_g}^T \frac{\mathbf{w}_{t+\Delta t}}{\Delta t} + \mathbf{w}_t^T\left(\mathcal{M}_{\mathcal{DT}} - \mathcal{L}_{\mathcal{FP}}\right)\right)^2 d\mathbf{x}$$

Residual Error$(e) =$

$$\mathbf{p_g}^T \frac{\mathbf{w}_{t+\Delta t}}{\Delta t} + \mathbf{w}_t^T\left(\mathcal{M}_{\mathcal{DT}} - \mathcal{L}_{\mathcal{FP}}\right)$$

$$\min_{w^i_{t+\Delta t}} \quad \mathbf{J} = \frac{1}{2}\int_V e^2(t,\mathbf{x})\, d\mathbf{x}$$

subject to: $\sum_{i=1}^{N} w^i_{t+\Delta t} = 1$

$w^i_{t+\Delta t} \geq 0, i = 1,2,\ldots,N$

$$\mathbf{J} = \frac{1}{2}\mathbf{w}^T_{t+\Delta t}\underbrace{\left(\int_V \frac{\mathbf{p_g}\mathbf{p_g}^T}{\Delta t^2} d\mathbf{x}\right)}_{\mathbf{M}}\mathbf{w}_{t+\Delta t}$$

$$+$$

$$\mathbf{w}^T_{t+\Delta t}\underbrace{\left(\int_V \frac{\mathbf{p_g}}{\Delta t}\left(\mathcal{M}_{\mathcal{DT}} - \mathcal{L}_{\mathcal{FP}}\right)^T d\mathbf{x}\right)}_{\mathbf{N}}\mathbf{w}_t$$

$$+$$

Terms containing $\mathbf{w}_t$

Convex Optimization Problem $\Rightarrow$

$$\min_{w^i_{t+\Delta t}} \quad \mathbf{J} = \frac{1}{2}\mathbf{w}^T_{t+\Delta t}\mathbf{M}\mathbf{w}_{t+\Delta} + \mathbf{w}^T_{t+\Delta t}\mathbf{N}\mathbf{w}_t$$

subject to: $\sum_{i=1}^{N} w^i_{t+\Delta t} = 1$

$w^i_{t+\Delta t} \geq 0, i = 1,2,\ldots,N$

$$\min_{w^i_{t+\Delta t}} \quad \mathbf{J} = \frac{1}{2}\mathbf{w}^T_{t+\Delta t}\mathbf{M}\mathbf{w}_{t+\Delta} + \mathbf{w}^T_{t+\Delta t}\mathbf{N}\mathbf{w}_t$$

subject to: $\sum_{i=1}^{N} w^i_{t+\Delta t} = 1$

$$w^i_{t+\Delta t} \geq 0, i = 1, 2, \ldots, N$$

$$\mathbf{M} = \left(\int_V \frac{\mathbf{p_g}\mathbf{p_g}^T}{\Delta t^2} d\mathbf{x} \right)$$

Integral of product of two Gaussians:

$$\mathcal{N}\left(\mathbf{x}|\boldsymbol{\mu}^i, \boldsymbol{\Sigma}^i\right) \times \mathcal{N}\left(\mathbf{x}|\boldsymbol{\mu}^j, \boldsymbol{\Sigma}^j\right)$$

We get closed form!!!

$$M_{ij} = \frac{1}{\Delta t^2} \left| 2\pi \left(\boldsymbol{\Sigma}^i + \boldsymbol{\Sigma}^j \right) \right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2} \left(\boldsymbol{\mu}^i - \boldsymbol{\mu}^j \right)^T \left(\boldsymbol{\Sigma}^i + \boldsymbol{\Sigma}^j \right)^{-1} \left(\boldsymbol{\mu}^i - \boldsymbol{\mu}^j \right) \right]$$

$$\min_{w^i_{t+\Delta t}} \quad \mathbf{J} = \frac{1}{2}\mathbf{w}^T_{t+\Delta t}\mathbf{M}\mathbf{w}_{t+\Delta} + \mathbf{w}^T_{t+\Delta t}\mathbf{N}\mathbf{w}_t$$

subject to: $\sum_{i=1}^{N} w^i_{t+\Delta t} = 1$

$$w^i_{t+\Delta t} \geq 0, i = 1, 2, \ldots, N$$

$$\mathbf{M} = \left(\int_V \frac{\mathbf{p_g}\mathbf{p_g}^T}{\Delta t^2} d\mathbf{x}\right)$$

Integral of product of two Gaussians:

$$\mathscr{N}\left(\mathbf{x}|\boldsymbol{\mu}^i, \boldsymbol{\Sigma}^i\right) \times \mathscr{N}\left(\mathbf{x}|\boldsymbol{\mu}^j, \boldsymbol{\Sigma}^j\right)$$

We get closed form!!!

$$M_{ij} = \frac{1}{\Delta t^2}\left|2\pi\left(\boldsymbol{\Sigma}^i + \boldsymbol{\Sigma}^j\right)\right|^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\boldsymbol{\mu}^i - \boldsymbol{\mu}^j\right)^T\left(\boldsymbol{\Sigma}^i + \boldsymbol{\Sigma}^j\right)^{-1}\left(\boldsymbol{\mu}^i - \boldsymbol{\mu}^j\right)\right]$$

$$\mathbf{N} = \left(\int_V \frac{\mathbf{p_g}}{\Delta t}\left(\mathscr{M}_{\mathscr{D}\mathscr{T}} - \mathscr{L}_{\mathscr{F}\mathscr{P}}\right)^T d\mathbf{x}\right)$$

$$\mathscr{M}_{\mathscr{D}\mathscr{T}} = \sum_{i=1}^{N}\left(\frac{\partial p^T_{g_i}}{\partial \boldsymbol{\mu}^i_t}\dot{\boldsymbol{\mu}}^i_t + \mathrm{Tr}\left[\frac{\partial p_{g_i}}{\partial \boldsymbol{\Sigma}^i_t}\dot{\boldsymbol{\Sigma}}^i_t\right] - \frac{p_{g_i}}{\Delta t}\right)$$

$$\mathscr{L}_{\mathscr{F}\mathscr{P}} = -\sum_{l=1}^{N}\mathbf{f}^T\frac{\partial p_{g_l}}{\partial \mathbf{x}} - p_{g_l}\mathrm{tr}\frac{\partial \mathbf{f}}{\partial \mathbf{x}} + \frac{1}{2}\sum_{l=1}^{N}\mathrm{tr}\left[\sum_{i,j=1}^{n}\frac{\partial^2\left[\left(\mathbf{g}\mathbf{Q}\mathbf{g}^T\right)_{ij}p_{g_l}\right]}{\partial x_i \partial x_j}\right]$$

$$n_{ij} = \frac{1}{\Delta t}\int_V p_{g_i}\frac{\partial p^T_{g_j}}{\partial \boldsymbol{\mu}^j}\dot{\boldsymbol{\mu}}^j + p_{g_i}\mathrm{Tr}\left[\frac{\partial p_{g_j}}{\partial \mathbf{P}^j}\dot{\mathbf{P}}^j\right] - \frac{p_{g_i}}{\Delta t}p_{g_j} + p_{g_i}\frac{\partial p_{g_j}}{\partial \mathbf{x}}\mathbf{f}(t,\mathbf{x}) + p_{g_i}p_{g_j}\mathrm{Tr}\left[\frac{\partial \mathbf{f}(t,\mathbf{x})}{\partial \mathbf{x}}\right] - p_{g_i}\frac{1}{2}\sum_{l=1}^{N}\mathrm{tr}\left[\sum_{i,j=1}^{n}\frac{\partial^2\left[\left(\mathbf{g}\mathbf{Q}\mathbf{g}^T\right)_{ij}p_{g_l}\right]}{\partial x_i \partial x_j}\right] d\mathbf{x}$$

EQUATION OF SYSTEM: DISCRETE TIME

$$\mathbf{x}_{k+1} = \boldsymbol{\phi}(k, \mathbf{x}_k) + \boldsymbol{\eta}_k, \ \boldsymbol{\eta}_k = \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$$

pdf at $t = 0$ is $p(t_0, \mathbf{x}_0)$. Find pdf $p(t_k, \mathbf{x})$ at any given time $t = t_k > 0$, while taking into consideration the solution of the CKE.

CKE: $$p(t_{k+1}, \mathbf{x}_{k+1}) = \int p(t_{k+1}, \mathbf{x}_{k+1} | t_k, \mathbf{x}_k) p(t_k, \mathbf{x}_k) d\mathbf{x}_k$$

Gaussian Mixture Approximation: $$p(t_{k+1}, \mathbf{x}_{k+1}) = \hat{p}(t_{k+1}, \mathbf{x}_{k+1}) + \text{Residual error}(e)$$

$$\hat{p}(t_{k+1}, \mathbf{x}_{k+1}) = \sum_{i=1}^{N} w_{k+1}^i \mathcal{N}(\mathbf{x}_{k+1} | \boldsymbol{\mu}_{k+1}^i, \mathbf{P}_{k+1}^i)$$

Constraints: $$\sum_{i=1}^{N} w_{t=k}^i = 1; \quad w_{t=k}^i \geq 0, \forall i$$

- Each Gaussian kernel capture the local behavior of non-Gaussian density function.

$$\boldsymbol{\mu}_{k+1}^i = \boldsymbol{\phi}(t_k \boldsymbol{\mu}_k^i)$$

$$\mathbf{P}_{k+1}^i = \mathbf{F}\mathbf{P}_k^i + \mathbf{P}_k^i \mathbf{F}^T + \mathbf{Q}_k, \ \mathbf{F} = \frac{\partial \boldsymbol{\phi}(k, \mathbf{x}_k)}{\partial \mathbf{x}_k}|_{\mathbf{x}_k = \boldsymbol{\mu}_k^i}$$

- Update the weights of Gaussian Sum Mixture such that following residual error is minimized over the whole domain. [7] [8]
  - residual error$(e) = p(t_{k+1}, \mathbf{x}_{k+1}) - \hat{p}(t_{k+1}, \mathbf{x}_{k+1})$.

$$\text{min:} \quad \frac{1}{2}\int |p(t_{k+1}, \mathbf{x}_{k+1}) - \hat{p}(t_{k+1}, \mathbf{x}_{k+1})|^2 d\mathbf{x}_{k+1}$$

$$\text{subject to:} \sum_{i=1}^{N} w_{k+1}^i = 1 \qquad w_{k+1}^i \geq 0$$

[7] Terejanu et. al., "Uncertainty propagation for nonlinear dynamic systems using Gaussian mixture models", JGCD, 2008

[8] Terejanu et. al., "Adaptive Gaussian Sum Filter for Nonlinear Bayesian Estimation", TAC, 2011

Minimizing the residual error in the least square sense(Terejanu et. al.[9]):

$$\min: \quad \frac{1}{2}\int |p(t_{k+1},\mathbf{x}_{k+1}) - \hat{p}(t_{k+1},\mathbf{x}_{k+1})|^2 d\mathbf{x}_{k+1}$$

$$\text{subject to: } \sum_{i=1}^{N} w_{k+1}^i = 1 \qquad w_{k+1}^i \geq 0$$

Expanding the terms we get:

$$\frac{1}{2}\int |p(t_{k+1},\mathbf{x}_{k+1}) - \hat{p}(t_{k+1},\mathbf{x}_{k+1})|^2 d\mathbf{x}_{k+1}$$
$$= \underbrace{\frac{1}{2}\int |p(t_{k+1},\mathbf{x}_{k+1})|^2 d\mathbf{x}_{k+1}}_{\text{constant}} - \int |p(t_{k+1},\mathbf{x}_{k+1})\hat{p}(t_{k+1},\mathbf{x}_{k+1})| d\mathbf{x}_{k+1}$$
$$+ \frac{1}{2}\int |\hat{p}(t_{k+1},\mathbf{x}_{k+1})|^2 d\mathbf{x}_{k+1}$$

[9] Terejanu et al, JGCD 2008, TAC 2011

Ignoring constant,

$$J = \frac{1}{2}\int |\hat{p}(t_{k+1},\mathbf{x}_{k+1})|^2 d\mathbf{x}_{k+1} - \int |p(t_{k+1},\mathbf{x}_{k+1})\hat{p}(t_{k+1},\mathbf{x}_{k+1})| d\mathbf{x}_{k+1}$$

Substituting $\hat{p}(t_{k+1},\mathbf{x}_{k+1}) = \sum_{i=1}^{N} w_{k+1}^i \mathcal{N}(\mathbf{x}_{k+1}|\boldsymbol{\mu}_{k+1}^i,\mathbf{P}_{k+1}^i)$, we get,

$$\begin{aligned} J &= \frac{1}{2}\int |\sum_{i=1}^{N} w_{k+1}^i \mathcal{N}(\mathbf{x}_{k+1}|\boldsymbol{\mu}_{k+1}^i,\mathbf{P}_{k+1}^i)|^2 d\mathbf{x}_{k+1} - \int \sum_{i=1}^{N} w_{k+1}^i \mathcal{N}(\mathbf{x}_{k+1}|\boldsymbol{\mu}_{k+1}^i,\mathbf{P}_{k+1}^i) p(t_{k+1},\mathbf{x}_{k+1}) d\mathbf{x}_{k+1} \\ &= \frac{1}{2}\mathbf{w}_{k+1}^T \mathbf{M}\mathbf{w}_{k+1} - \mathbf{w}_{k+1}^T \mathbf{y} \end{aligned}$$

$$\begin{aligned} m_{ij} &= \int \mathcal{N}(\mathbf{x}_{k+1}|\boldsymbol{\mu}_{k+1}^i,\mathbf{P}_{k+1}^i)\mathcal{N}(\mathbf{x}_{k+1}|\boldsymbol{\mu}_{k+1}^j,\mathbf{P}_{k+1}^j) d\mathbf{x}_{k+1} \\ &= |2\pi\left(\mathbf{P}_{k+1}^i + \mathbf{P}_{k+1}^j\right)|^{-1/2} exp[-\frac{1}{2}(\boldsymbol{\mu}_{k+1}^i - \boldsymbol{\mu}_{k+1}^j)^T(\mathbf{P}_{k+1}^i + \mathbf{P}_{k+1}^j)^{-1}(\boldsymbol{\mu}_{k+1}^i - \boldsymbol{\mu}_{k+1}^j)] \end{aligned}$$

$$y_i = \int p(t_{k+1},\mathbf{x}_{k+1})\mathcal{N}(\mathbf{x}_{k+1}|\boldsymbol{\mu}_{k+1}^i,\mathbf{P}_{k+1}^i) d\mathbf{x}_{k+1}$$

$$y_i = \int p(t_{k+1}, \mathbf{x}_{k+1}) \mathcal{N}(\mathbf{x}_{k+1} | \boldsymbol{\mu}^i_{k+1}, \mathbf{P}^i_{k+1}) d\mathbf{x}_{k+1}$$

Using CKE, replace $p(t_{k+1}, \mathbf{x}_{k+1})$.

$$y_i = \int \mathcal{N}(\mathbf{x}_{k+1} | \boldsymbol{\mu}^i_{k+1}, \mathbf{P}^i_{k+1}) d\mathbf{x}_{k+1} \underbrace{\int p(t_{k+1}, \mathbf{x}_{k+1} | t_k, \mathbf{x}_k) p(t_k, \mathbf{x}_k) d\mathbf{x}_k}_{\text{CKE}}$$

For the given system , $p(t_{k+1}, \mathbf{x}_{k+1} | t_k, \mathbf{x}_k) = \mathcal{N}(\mathbf{x}_{k+1} | \boldsymbol{\phi}(k, \mathbf{x}_k), \mathbf{Q}_k)$.

$$\begin{aligned} y_i &= \int\int \mathcal{N}(\mathbf{x}_{k+1} | \boldsymbol{\mu}^i_{k+1}, \mathbf{P}^i_{k+1}) \mathcal{N}(\mathbf{x}_{k+1} | \boldsymbol{\phi}(k, \mathbf{x}_k), \mathbf{Q}_k) p(t_k, \mathbf{x}_k) d\mathbf{x}_k d\mathbf{x}_{k+1} \\ &= \int p(t_k, \mathbf{x}_k) \left[\underbrace{\int \mathcal{N}(\mathbf{x}_{k+1} | \boldsymbol{\mu}^i_{k+1}, \mathbf{P}^i_{k+1}) \mathcal{N}(\mathbf{x}_{k+1} | \boldsymbol{\phi}(k, \mathbf{x}_k), \mathbf{Q}_k) d\mathbf{x}_{k+1}}_{\text{Integral of product of two Gaussian}} \right] d\mathbf{x}_k \\ &= \int p(t_k, \mathbf{x}_k) \mathcal{N}(\boldsymbol{\phi}(k, \mathbf{x}_k) | \boldsymbol{\mu}^i_{k+1}, \mathbf{P}^i_{k+1} + \mathbf{Q}_k) d\mathbf{x}_k \end{aligned}$$

$$y_i = \int p(t_k, \mathbf{x}_k) \mathcal{N}(\boldsymbol{\phi}(k, \mathbf{x}_k) | \boldsymbol{\mu}^i_{k+1}, \mathbf{P}^i_{k+1} + \mathbf{Q}_k) d\mathbf{x}_k$$

Replace $p(t_k, \mathbf{x}_k)$ with $\hat{p}(t_k, \mathbf{x}_k) = \sum_{i=1}^N w^i_k \mathcal{N}(\mathbf{x}_k | \boldsymbol{\mu}^i_k, \mathbf{P}^i_k)$

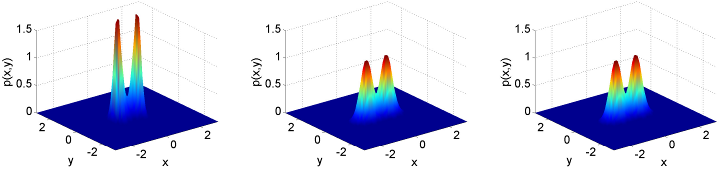
$$\begin{aligned} y_i &= \sum_{j=1}^N w^j_k \int \mathcal{N}(\mathbf{x}_k | \boldsymbol{\mu}^j_k, \mathbf{P}^j_k) \mathcal{N}(\boldsymbol{\phi}(k, \mathbf{x}_k) | \boldsymbol{\mu}^i_{k+1}, \mathbf{P}^i_{k+1} + \mathbf{Q}_k) d\mathbf{x}_k \\ &= \sum_{j=1}^N w^j_k n_{ij} \end{aligned}$$

$$n_{ij} = \int \mathcal{N}(\mathbf{f}(k, \mathbf{x}_k) | \boldsymbol{\mu}^i_{k+1}, \mathbf{P}^i_{k+1} + \mathbf{Q}_k) \mathcal{N}(\mathbf{x}_k | \boldsymbol{\mu}^j_k, \mathbf{P}^j_k) d\mathbf{x}_k$$

$$\min_{\mathbf{w}_{k+1}} \quad J = \frac{1}{2} \mathbf{w}^T_{k+1} \mathbf{M} \mathbf{w}_{k+1} - \mathbf{w}^T_{k+1} \mathbf{N} \mathbf{w}_k$$

$$\text{subject to:} \quad \mathbf{1}^T \mathbf{w}_{k+1} = 1 \qquad \mathbf{w}_{k+1} \geq 0$$

- Duffing Oscillator with a soft spring:
  $\ddot{x} + \eta\dot{x} + \alpha x + \beta x^3 = g(t)\mathscr{G}(t)$
    - $g(t) = 1,\ Q = 1,\ \eta = 10,\ \alpha = -1,\ \beta = 3.$
    - Stationary pdf: $p(x, \dot{x}) \propto \exp\left(-2\frac{\eta}{g^2 Q}\left(\frac{\alpha}{2}x^2 + \frac{\beta}{4}x^4 + \frac{1}{2}\dot{x}^2\right)\right)$

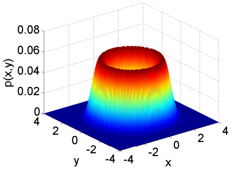


(a) True pdf (b) FPKE Solution (c) CKE Solution

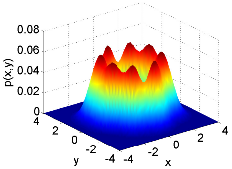
FPKE/CKE

Example: Vanderpol Oscillator

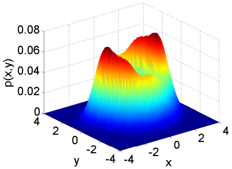
- Vanderpol Oscillator : $\ddot{x}+\beta\dot{x}+x+\alpha(x^2+\dot{x}^2)\dot{x}=g(t)\mathscr{G}(t)$
  - $g(t)=1$, $Q=1/\pi$, $\alpha=0.125$, $\beta=-0.5$.
  - Stationary pdf: $p(x,\dot{x})\propto\exp\left(-\frac{\eta}{2g^2}\left(\beta(x^2+\dot{x}^2)+\frac{\alpha}{2}(x^2+\dot{x}^2)^2\right)\right)$



(d) True pdf



(e) FPKE Solution



(f) CKE Solution

FPKE/CKE

EXAMPLE: LORENZ SYSTEM

Consider the following system:

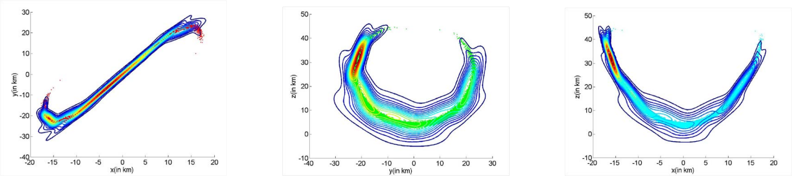
$$\dot{x} = \alpha\left(-x+y\right)$$

$$\dot{y} = \beta x - y - xz$$

$$\dot{z} = -\gamma z + xy \qquad \alpha = 10 \quad \beta = 28 \quad \gamma = \frac{20}{3}$$

$$p\left(x,y,z,t_0\right) = \mathcal{N}\left(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0\right)$$

$$\boldsymbol{\mu} = \begin{bmatrix} -0.16 & -0.16 & 8 \end{bmatrix} \qquad \boldsymbol{\Sigma} = \mathrm{diag}\left(1,1,1\right)$$



(g) XY Plane (h) YZ Plane (i) XZ Plane

FIGURE: contours after 0.037 sec

The equations of motion of a satellite with atmospheric drag & J_2 effect is[10]:

$$\begin{aligned}
\ddot{x} + \frac{\mu x}{r^3} &= J_{2_x} + a_{D_x}, & J_{2_x} &= -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(1 - 5\frac{z^2}{r^2}\right)\frac{x}{r} \\
\ddot{y} + \frac{\mu y}{r^3} &= J_{2_y} + a_{D_y}, & J_{2_y} &= -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(1 - 5\frac{z^2}{r^2}\right)\frac{y}{r} \\
\ddot{z} + \frac{\mu z}{r^3} &= J_{2_z} + a_{D_z}, & J_{2_z} &= -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(3 - 5\frac{z^2}{r^2}\right)\frac{z}{r}
\end{aligned} \tag{2}$$

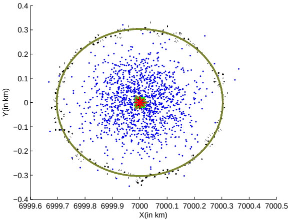
a_{D_x}, a_{D_y}, and a_{D_z} denote the atmospheric drag forces modeled as:

$$\begin{aligned}
a_{D_x} &= -\frac{1}{2}B\rho\,(\dot{x} + \Omega_e y)\sqrt{(\dot{x} + \Omega_e y)^2 + (\dot{y} - \Omega_e x)^2 + (\dot{z})^2} \\
a_{D_y} &= -\frac{1}{2}B\rho\,(\dot{y} - \Omega_e x)\sqrt{(\dot{x} + \Omega_e y)^2 + (\dot{y} - \Omega_e x)^2 + (\dot{z})^2} \\
a_{D_z} &= -\frac{1}{2}B\rho\dot{z}\sqrt{(\dot{x} + \Omega_e y)^2 + (\dot{y} - \Omega_e x)^2 + (\dot{z})^2}
\end{aligned} \tag{3}$$

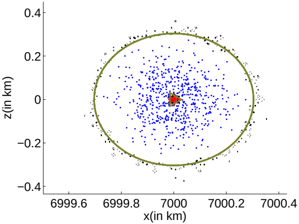
Initial Conditions: $p(t_0, \underbrace{x, y, z, \dot{x}, \dot{y}, \dot{z}}_{\mathbf{x}}) = \mathcal{N}(\mathbf{x}; \mu_0, \mathbf{P}_0)$, $\mu_0 = \left\{\underbrace{7\times 10^3, 0, 0}_{km}, \underbrace{0, -1.0374, 7.4771}_{km/sec}\right\}^T$,

$$\mathbf{P}_0 = diag\left(\underbrace{0.01, 0.01, 0.01}_{km^2}, \underbrace{10^{-6}, 10^{-6}, 10^{-6}}_{km^2/sec^2}\right)^T$$

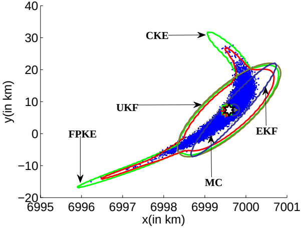
[10] Kumar et. al., "Nonlinear Uncertainty Propagation for Perturbed Two-Body Orbits", JGCD, 2014



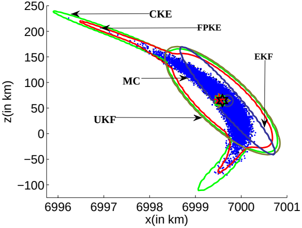
(a) pdf contours in XY plane at t = 0



(b) pdf contours in XZ plane at t = 0

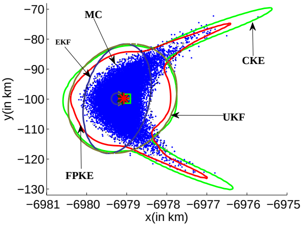


(c) pdf contours in XY plane at 3.2 hr

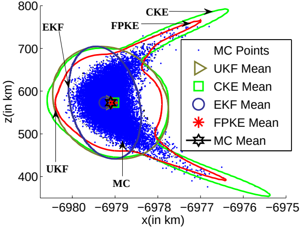


(d) pdf contours in XZ plane at 3.2 hr

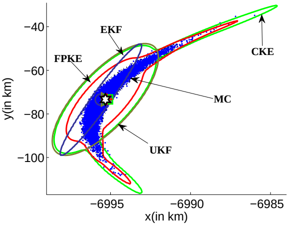
Adaptive Gaussian Mixture Model



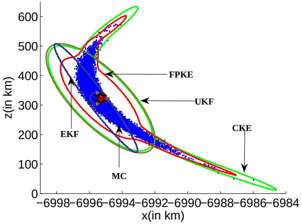
(a) pdf contours in XY plane at 4 hr



(b) pdf contours in XZ plane at 4 hr



(c) pdf contours in XY plane at 5.6585 hr



(d) pdf contours in XZ plane at 5.6585 hr