SPECTRAL METHODS

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- Polynomial chaos is a term coined by Norbert Wiener in 1938.
- The basic idea of this approach is to approximate the stochastic system state in terms of finite-dimensional series expansion in the stochastic space.
- The completeness of the space allows for the accurate representation of any PDF using a suitable basis.
  - Certain bases can be chosen to represent given PDF with the fewest number of terms.
  - For example, the Legendre polynomial can be used to represent the Uniform distribution with only two terms.
- The unknown coefficients are determined by minimizing an appropriate norm of the residual.

$\dot{\mathbf{x}}(t,\theta) = \mathbf{A}(\theta)\mathbf{x}(t,\theta) + \mathbf{B}(\theta)\mathbf{u}(t)$, $\mathbf{x}(t,\theta) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^r$, $\theta \in \mathbb{R}^m$

- p is a function of random variable ξ with known probability distribution function (pdf) $f(\xi)$, i.e., $\theta = \theta(\xi)$.
- Polynomial Chaos Representation:

$$x_i(t,\theta) = \sum_{k=1}^{N} x_{i_k}(t)\phi_k(\xi) = \mathbf{x}_i^T(t)\Phi(\xi)$$

$$A_{ij}(\theta) = \sum_{k=1}^{N} a_{ij_k}\phi_k(\xi) = \mathbf{a}_{ij}^T\Phi(\xi)$$

$$B_{ij}(\theta) = \sum_{k=1}^{N} b_{ij_k}\phi_k(\xi) = \mathbf{b}_{ij}^T\Phi(\xi)$$

- $\Phi(.) \in \mathbb{R}^N$ is a vector of polynomials basis functions orthogonal to the pdf $f(\xi)$.
  - *Gram-Schmidt Orthogonalization Process*

PC Coefficients

- *Normal Equations:*

$$a_{ij_k} = \frac{\langle A_{ij}(\theta(\xi)), \phi_k(\xi)\rangle}{\langle \phi_k(\xi), \phi_k(\xi)\rangle}, \; b_{ij_k} = \frac{\langle B_{ij}(\theta(\xi)), \phi_k(\xi)\rangle}{\langle \phi_k(\xi), \phi_k(\xi)\rangle}$$

- $\langle u(\xi), v(\xi)\rangle = \int_{\mathbb{R}^m} u(\xi)v(\xi)f(\xi)d\xi$ represents the norm introduced by pdf $f(\xi)$.
- x_{i_k} are unknowns.

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Error Dynamics:

$$e_i(\xi) = \sum_{k=1}^{N} \dot{x}_{i_k}(t)\phi_k(\xi) - \sum_{j=1}^{n}\left(\sum_{k=1}^{N} a_{ij_k}\phi_k(\xi)\right)\left(\sum_{k=1}^{N} x_{j_k}(t)\phi_k(\xi)\right) + \sum_{j=1}^{m}\left(\sum_{k=1}^{N} b_{ij_k}\phi_k(\xi)\right)u_j, \quad i = 1,2,\cdots,n$$

Galerkin Projection:

$$\langle e_i(\xi), \phi_k(\xi)\rangle = 0, \quad i = 1,2,\cdots,n, \quad k = 1,2,\cdots,N$$

Galerkin Projection: $\langle e_i(\xi), \phi_k(\xi)\rangle = 0$

- *nN deterministic differential equations:*

$$\mathbf{M}\dot{\mathbf{c}}(t) = \mathbf{K}\mathbf{c}(t) + \mathbf{D}\mathbf{u}(t)$$

  where $\mathbf{c}(t) = \left\{\mathbf{x}_1^T(t), \mathbf{x}_2^T(t), \cdots, \mathbf{x}_n^T(t)\right\}^T$ is a vector of nN unknown coefficients.

- $\mathbf{M} \in \mathbb{R}^{nN\times nN}$, $\mathbf{K} \in \mathbb{R}^{nN\times nN}$ and $\mathbf{D} \in \mathbb{R}^{nN\times m}$ are given by

$$M_{kl} = \langle \phi_i(\xi), \phi_j(\xi)\rangle, \; k = (i-1)n+1, \cdots in, l = (j-1)n+1, \cdots jn$$

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- Note: $N = \frac{(d+m)!}{d!m!} - 1$, d being the degree of the polynomials.
- *PC projections leads to higher order deterministic linear system.*

Generalized Polynomial Chaos

Linear Systems: $\dot{\mathbf{x}}(t,\theta) = \mathbf{A}(\theta)\mathbf{x}(t,\theta) + \mathbf{B}(\theta)\mathbf{u}(t)$

Moments of State Variables,

$$x_i(t,\theta) = \sum_{k=1}^{N} x_{i_k}(t)\phi_k(\xi) = \mathbf{x}_i^T(t)\Phi(\xi)$$

- Mean = $\int_{\mathbb{R}^m} x_i(t,\xi)f(\xi)d\xi = x_{i_1}$.
- Variance = $\sum_{k=2}^{N} x_{i_k}^2 \langle \phi_k(\xi), \phi_k(\xi) \rangle$

PC REPRESENTATION

$$x_i(t,\theta) = \sum_{k=1}^{N} c_{i_k}(t)\phi_k(\xi) = \mathbf{c}_i^T(t)\Phi(\xi) \Rightarrow \mathbf{x}(t,\theta) = \mathbf{C}(t)\Phi(\xi)$$

$$\theta_i(\xi) = \sum_{k=1}^{N} a_{i_k}\phi_k(\xi) = \mathbf{a}_i^T\Phi(\xi) \Rightarrow \theta(\xi) = \mathbf{A}\Phi(\xi)$$

ERROR DYNAMICS

$$\mathbf{e}(\mathbf{C},\xi) = \mathbf{f}(\mathbf{A}\Phi(\xi),\mathbf{C}(t)\Phi(\xi)) + \mathbf{g}(\mathbf{A}\Phi(\xi),\mathbf{C}(t)\Phi(\xi))\mathbf{u}(t)$$

Galerkin Projection:

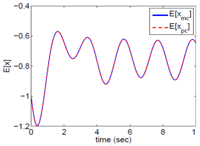
$$\langle e_i(\xi),\phi_k(\xi)\rangle = 0, \quad i = 1,2,\cdots,n, \quad k = 1,2,\cdots,N$$

Numerical quadrature methods are required to compute projection integrals!!

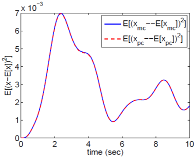
Generalized Polynomial Chaos

Duffing Oscilaltor: $\ddot{x}(t)+\eta\dot{x}+\alpha x+\beta x^3=\sin(3t)$, $\beta=2,\ \eta\in\mathscr{U}(0.9,1.4),\ \alpha\in\mathscr{U}(-1.45,-0.95)$

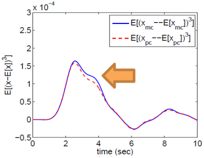
2nd Order gPC & Monte Carlo Solution:



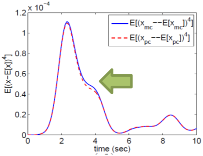
Mean



Variance

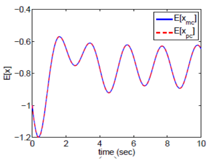


3rd Central Moment

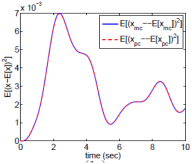


4th Central Moment

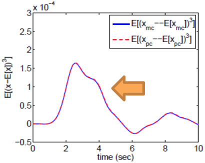
6th Order gPC & Monte Carlo Solution:



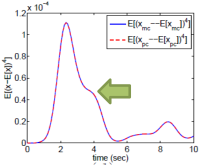
Mean



Variance



3rd Central Moment



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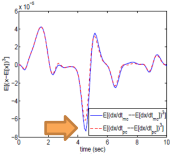
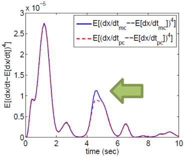
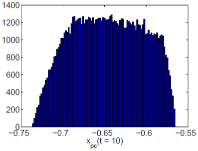


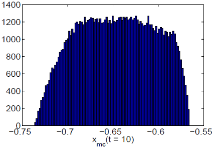


3^{rd} Central Moment

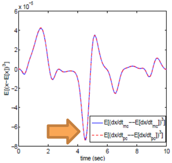
4^{th} Central Moment

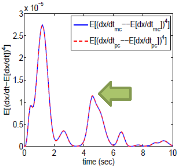
Histograms of x:





6^{th} Order gPC & Monte Carlo solution:

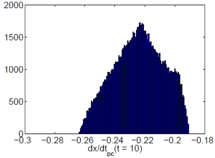


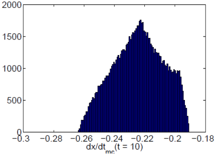


3^{rd} Central Moment

4^{th} Central Moment

Histograms of $\dot{x}$





- For complex dynamical systems, *the model equations are not explicitly available.*

PC Approximation:

$$x_i(t,\theta) = \sum_{k=1}^{N} c_{i_k}(t)\phi_k(\xi) = \mathbf{c}_i^T(t)\Phi(\xi) \Rightarrow \mathbf{x}(t,\theta) = \mathbf{C}(t)\Phi(\xi)$$

- Consider a least square performance index:

$$\min_{c_{i_k}} = \frac{1}{2}\int \mathbf{e}^T(t,\xi)\mathbf{e}(t,\xi)f(\xi)d\xi = \frac{1}{2}\langle \mathbf{e}(t,\xi), \mathbf{e}(t,\xi)\rangle >$$

where, $e_i(t,\xi) = x_i(t,\theta(\xi)) - \mathbf{c}_i^T(t)\Phi(\xi)$

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where, $e_i(t,\xi) = x_i(t,\theta(\xi)) - \mathbf{c}_i^T(t)\Phi(\xi)$

First-Order optimality condition: $\frac{\partial J}{\partial c_{i_k}} = 0$ leads to

$$\sum_{m=1}^{N} \langle \phi_l, \phi_m\rangle c_{i_m} = \langle x_i(t,\theta(\xi)), \phi_m(\xi)\rangle, \quad i = 1, 2, \cdots, n$$

$$\sum_{m=1}^{N} \langle \phi_l, \phi_m \rangle c_{i_m} = \langle x_i(t, \theta(\xi)), \phi_m(\xi) \rangle, \quad i = 1, 2, \cdots, n$$

- Linear Set of Equations, $\mathbf{MC} = \mathbf{b}$
- if $\phi_i(\xi)$ and $\phi_j(\xi)$ are orthogonal, then linear system is *decoupled*
- Quadrature Approximation:
  $\langle x_i(t, \theta(\xi)), \phi_m(\xi) \rangle \approx \sum_k w_k x_i(t, \theta(\xi_k)), \phi_m(\xi_k)$.

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- So, PC coefficients can be evaluated by Monte Carlo (MC) kind of simulations.
  - *MC points are replaced by quadrature points.*
  - *Computational cost can be greatly reduced by the use of non-product quadrature methods like CUT!!*

System Equations $\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t))$

- System flow: $\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t f(\tau, \mathbf{x}(\tau)) d\tau = \psi(t, \mathbf{x}_0)$.
- Perturbation trajectory:
  $\delta\mathbf{x}(t) = \psi(t, \mathbf{x}_0 + \delta\mathbf{x}_0) - \psi(t, \mathbf{x}_0) \approx \Phi(t, t_0)\delta\mathbf{x}_0, \;\; \Phi(t, t_0) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_0}$

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Taylor Series Expansion

- Higher order perturbation trajectory:

$$\delta\mathbf{x}(t) \approx \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_n=0}^{\infty} \frac{\delta x_{0_1}^{N_1}\delta x_{0_2}^{N_2}\cdots\delta x_{0_n}^{N_n}}{N_1!N_2!\cdots N_n!} \frac{\partial^{N_1+N_2+\cdots+N_n}}{\partial x_{0_1}^{N_1}\partial x_{0_2}^{N_2}\cdots\partial x_{0_n}^{N_n}} \psi(t, \mathbf{x}_0), \quad N_1 = N_2 = \cdots N_n \neq 0$$

- In other words, one can expand $\delta\mathbf{x}(t)$ in terms of polynomial basis functions:

$$\delta\mathbf{x}(t) \approx \sum_{i=1}^{N} c_i(t)\phi_i(\delta\mathbf{x}_0) = \mathbf{c}(t)\phi(\delta\mathbf{x}_0)$$

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- The coefficient of the linear term corresponds to $\Phi(t, t_0)$.
- Higher order coefficients have the meaning of *higher order state transition matrices.*
- They are valid only in the neighborhood of the nominal trajectory.

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If initial condition, $\mathbf{x}_0$ is a random variable with prescribed density function $f(\mathbf{x}_0)$, then it would make sense to compute the first order and high order state transition matrix valid over the domain of initial condition uncertainty.

- In this respect, *one can pose the following problem (also known as statistical linearization)* to compute state transition matrix equivalent coefficients, $c_i(t)$:

$$\min_{c_i(t)} J = \frac{1}{2}\int (\delta\mathbf{x}(t,\xi) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0))^T (\delta\mathbf{x}(t,\xi) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0))\,\rho(\mathbf{x}_0) d\delta\mathbf{x}_0$$
$$= \frac{1}{2}\langle(\delta\mathbf{x}(t,\xi) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0)), (\delta\mathbf{x}(t,\xi) - \mathbf{c}(t)\mathbf{p}(\delta\mathbf{x}_0))\rangle$$

- Similar to non-intrusive PC.
- *Derivative free approach to compute state transition matrices in a domain of interest!!*
  - Domain of interest is represented by initial state PDF.

$$\ddot{x} + \frac{\mu x}{r^3} = J_{2_x} + a_{D_x}, \qquad J_{2_x} = -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(1 - 5\frac{z^2}{r^2}\right)\frac{x}{r}$$

$$\ddot{y} + \frac{\mu y}{r^3} = J_{2_y} + a_{D_y}, \qquad J_{2_y} = -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(1 - 5\frac{z^2}{r^2}\right)\frac{y}{r}$$

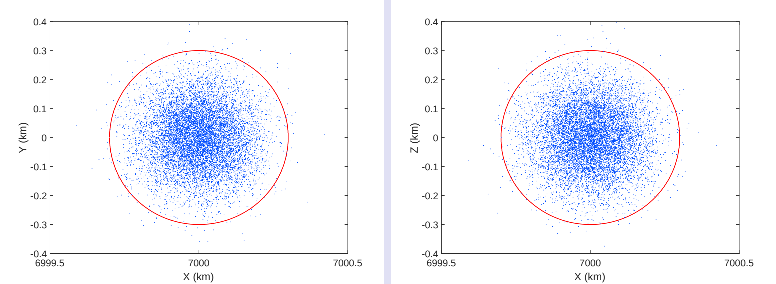
$$\ddot{z} + \frac{\mu z}{r^3} = J_{2_z} + a_{D_z}, \qquad J_{2_z} = -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(3 - 5\frac{z^2}{r^2}\right)\frac{z}{r}$$

Initial Conditions $\mathcal{N}(\mu_0, P_0)$

$$\mu_0 = \Big[\underbrace{7\times 10^3 \quad 0 \quad 0}_{\text{km}} \quad \underbrace{0 \quad -1.0374 \quad 7.4771}_{\text{km/s}}\Big]$$

$$\mathbf{P}_0 = \text{diag}\left(\underbrace{0.01 \quad 0.01 \quad 0.01}_{km^2} \quad \underbrace{0.000001 \quad 0.000001 \quad 0.000001}_{km^2/s^2}\right)$$

INITIAL CONDITION DOMAIN

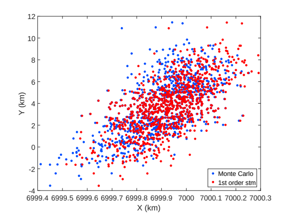


(a) X-Y plane (b) X-Z plane

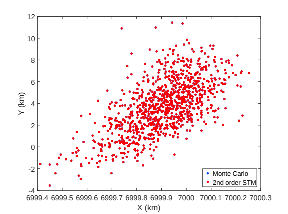
FIGURE: 10,000 Initial Conditions in X-Y plane and X-Z plane

745 CUT Points were used to compute higher order STM or PC coefficients.

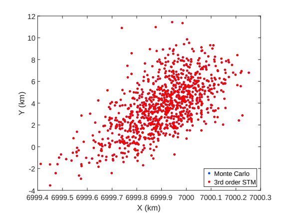
1st orbit period



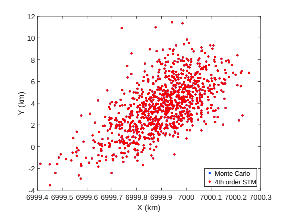
(a) 1st order STM



(b) 2nd order STM

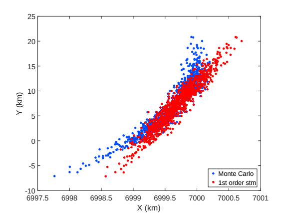


(c) 3rd order STM



(d) 4th order STM

2ND ORBIT PERIOD



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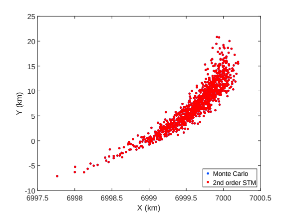

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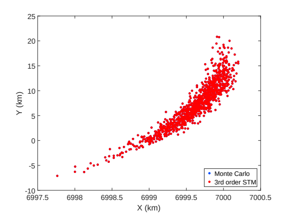

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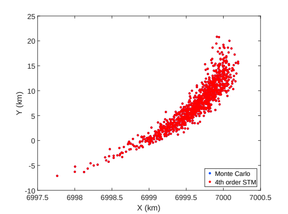

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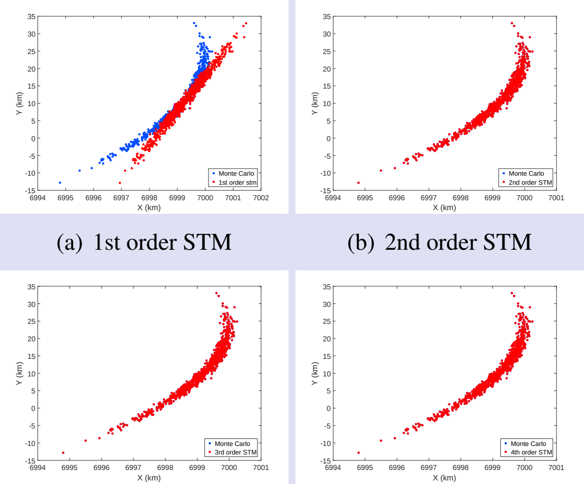
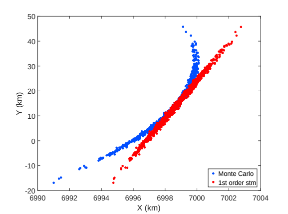
3rd orbit period



(a) 1st order STM (b) 2nd order STM

(c) 3rd order STM (d) 4th order STM

4TH ORBIT PERIOD



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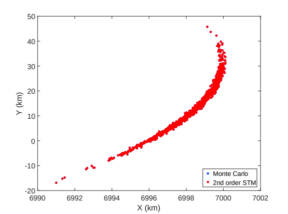
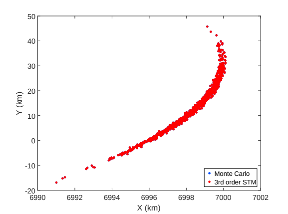
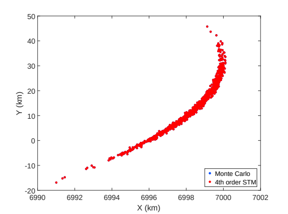


(b) 2nd order STM

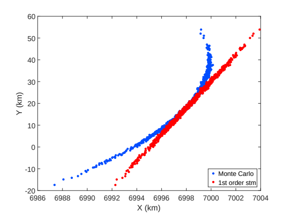


(c) 3rd order STM

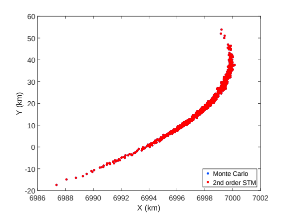


(d) 4th order STM

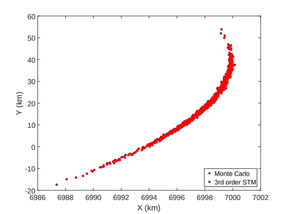
5th orbit period



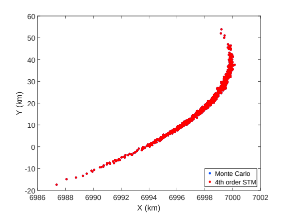
(a) 1st order STM



(b) 2nd order STM

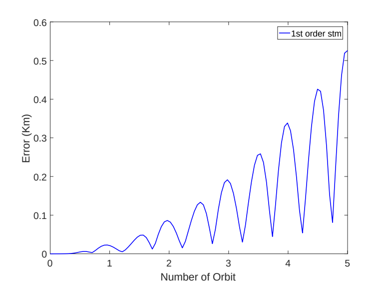


(c) 3rd order STM

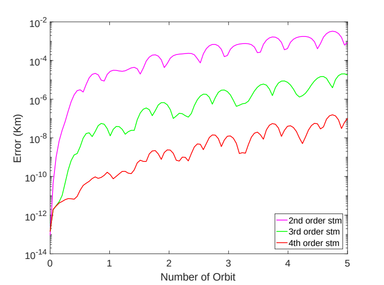


(d) 4th order STM

ERROR ANALYSIS



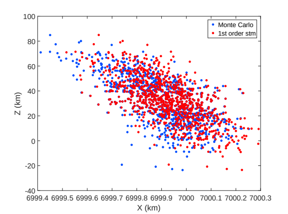
(a) 2-norm Error by 1st order STM



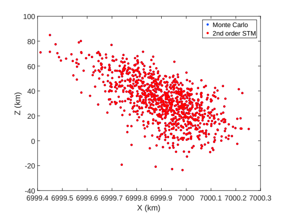
(b) 2-norm Error by higher order STM

FIGURE: 2-Norm Error Average over 10,000 Initial Conditions for Different Order STM Approximation

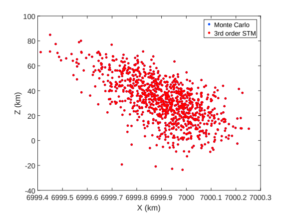
1st orbit period



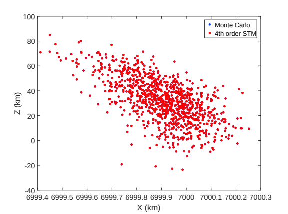
(a) 1st order STM



(b) 2nd order STM

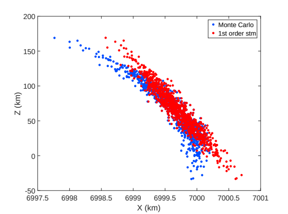


(c) 3rd order STM

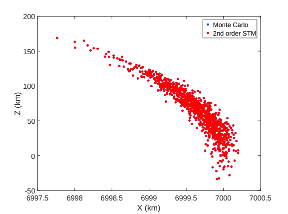


(d) 4th order STM

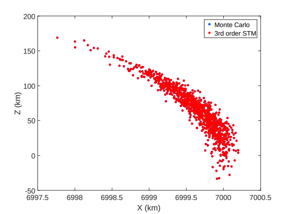
2nd orbit period



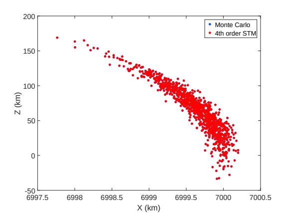
(a) 1st order STM



(b) 2nd order STM

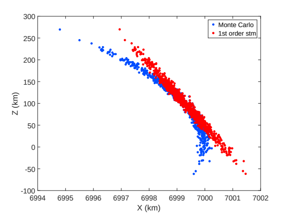


(c) 3rd order STM

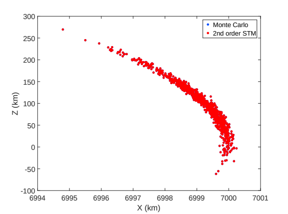


(d) 4th order STM

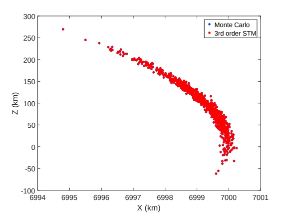
3rd orbit period



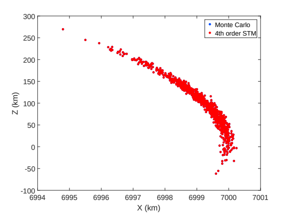
(a) 1st order STM



(b) 2nd order STM

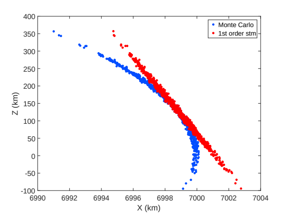


(c) 3rd order STM

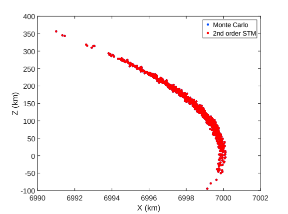


(d) 4th order STM

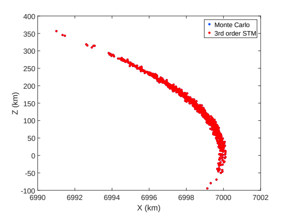
4th orbit period



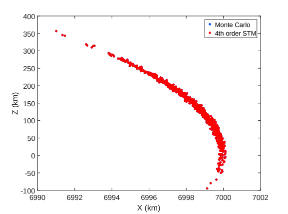
(a) 1st order STM



(b) 2nd order STM

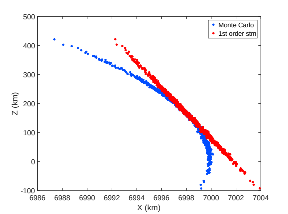


(c) 3rd order STM

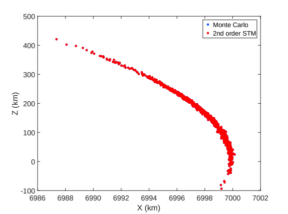


(d) 4th order STM

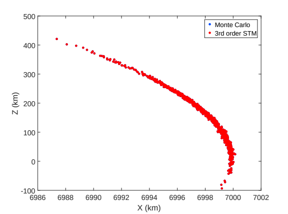
5th Orbit Period



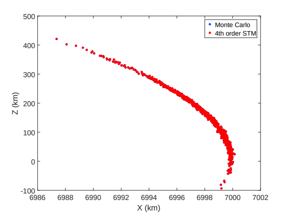
(a) 1st order STM



(b) 2nd order STM



(c) 3rd order STM



(d) 4th order STM

$\dot{\mathbf{x}} = \mathbf{A}(\Theta)\mathbf{x} + \mathbf{B}(\theta)\mathbf{u} + \mathscr{G}(\theta)\eta, \; \mathbf{x}(t_0) = \mu_0$

- The Gaussian white noise process, η is assumed to be uncorrelated in time and with other uncertainties in model parameters and initial conditions.
- The uncertain parameter vector θ is assumed to be a function of random vector ξ.
- Study the time-evolution of the state pdf: $p(t, \mathbf{x})$.

$\dot{\mathbf{x}} = \mathbf{A}(\Theta)\mathbf{x} + \mathbf{B}(\theta)\mathbf{u} + \mathscr{G}(\theta)\eta, \; \mathbf{x}(t_0) = \mu_0$

- For any particular realization of θ, the state pdf of is Gaussian for Gaussian initial conditions.
- Similarly, for any particular realization of η, the state uncertainty of can be efficiently characterized using a PC series expansion of the states.

1

[1] U. Konda, P. Singla, T. Singh, and P. D. Scott, "State uncertainty propagation in the presence of parametric uncertainty and additive white noise," ASME Journal of Dynamic Systems, Measurement, and Control 133, no. 5 (2011).

$\dot{\mathbf{x}} = \mathbf{A}(\Theta)\mathbf{x} + \mathbf{B}(\theta)\mathbf{u} + \mathscr{G}(\theta)\eta,\ \mathbf{x}(t_0) = \mu_0$

- For any particular realization of θ, the state pdf of is Gaussian for Gaussian initial conditions.
- Similarly, for any particular realization of η, the state uncertainty of can be efficiently characterized using a PC series expansion of the states.

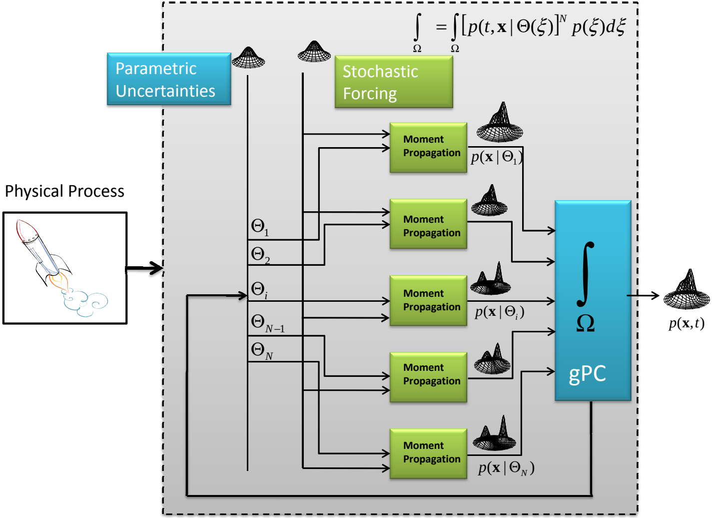
UNCERTAINTY MARRIAGE

Compute the conditional distribution function first and then determine the posterior distribution of the states.

[1]

[1] U. Konda, P. Singla, T. Singh, and P. D. Scott, "State uncertainty propagation in the presence of parametric uncertainty and additive white noise," ASME Journal of Dynamic Systems, Measurement, and Control 133, no. 5 (2011).

METHOD 1: CONDITIONING ON UNCERTAIN PARAMETER



$\dot{\mathbf{x}} = \mathbf{A}(\Theta)\mathbf{x} + \mathbf{B}(\theta)\mathbf{u} + \mathscr{G}(\theta)\eta$

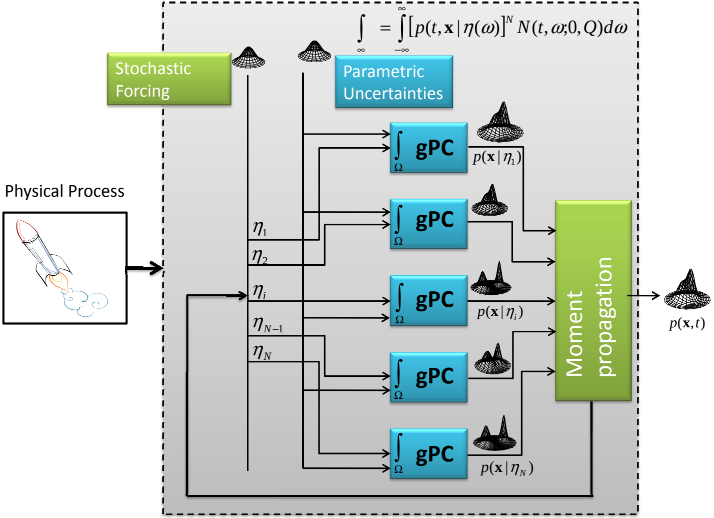
- Conditional state pdf, $p(x|\Theta) = \mathcal{N}(t, \mathbf{x}; \mu(t, \Theta)$

$$\dot{\mu} = \mathbf{A}(\Theta)\mu + \mathbf{B}\mathbf{u}$$
$$\dot{\Sigma} = \mathbf{A}(\Theta)\Sigma + \Sigma\mathbf{A}^T(\Theta) + \mathscr{G}(\Theta)\mathbf{Q}\mathscr{G}^T(\Theta)$$

- The state pdf can be computed as:

$$p(t,\mathbf{x}) = \int_{\Omega} p(t,\mathbf{x}|\Theta(\xi))p(\xi)d\xi$$
$$= \int_{\Omega} \mathcal{N}(t,\mathbf{x};\mu(t,\Theta),\Sigma(t,\Theta))p(\xi)d\xi$$

- gPC can be used to expand μ and Σ as a function of Θ.
- Equivalent to Multiple Model Approach.



Stochastic Forcing
Parametric Uncertainties
Physical Process
gPC
Moment propagation
$p(\mathbf{x}|\eta_1)$
$p(\mathbf{x}|\eta_i)$
$p(\mathbf{x}|\eta_N)$
$p(\mathbf{x},t)$
η_1
η_2
η_i
η_{N-1}
η_N

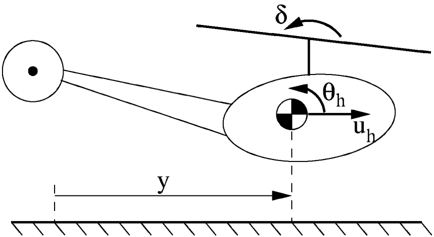
$\dot{\mathbf{x}} = \mathbf{A}(\Theta)\mathbf{x} + \mathbf{B}(\theta)\mathbf{u} + \mathscr{G}(\theta)\eta$

- gPC series characterizing, $p(x|\eta)$, results in a *linear system driven by Gaussian white noise:*

$$x_i(t) = \sum_{r=0}^{P} x_{i_r}(t,\omega)\phi_r(\xi) = \mathbf{x}_i^T(t,\omega)\Phi(\xi)$$
$$\dot{\mathbf{c}} = \mathbf{A}_p\mathbf{c} + \mathbf{B}_p\mathbf{u} + \mathscr{G}_p\eta(\omega)$$

- The distribution of PC coefficients is Gaussian with following mean and covariance:

$$\dot{\mu}_c = \mathbf{A}_p\mu_c + \mathbf{B}_p\mathbf{u} \quad (1)$$
$$\dot{\Sigma}_c = \mathbf{A}_p\Sigma_c + \Sigma_c\mathbf{A}_p^T + \mathscr{G}_p\mathbf{Q}\mathscr{G}_p^T \quad (2)$$



where,

$$\dot{\mathbf{x}} = A\mathbf{x} + B\delta + B_w u_w$$

$$\mathbf{x} = \begin{bmatrix} u_h & q_h & \theta_h & y \end{bmatrix}^T$$

$$A = \begin{bmatrix} p_1 & p_2 & -g & 0 \\ p_3 & p_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} p_5 & p_6 & 0 & 0 \end{bmatrix}^T$$

$$B_w = \begin{bmatrix} -p_1 & -p_3 & 0 & 0 \end{bmatrix}^T$$

$$\delta = -Kx$$

u_h: horizontal velocity (ft/s)

θ_h: pitch angle of the fuselage (centi-rad)

q_h: its derivative (centi-rad/s)

y : perturbation from a ground point reference (ft)

δ: longitudinal stick deflection (control effort)

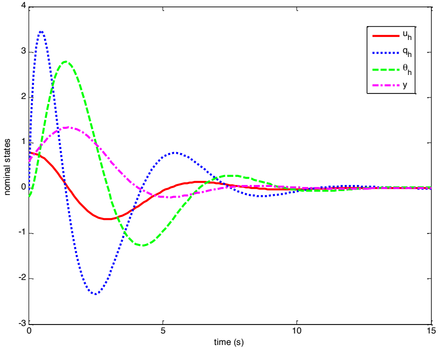
Aerodynamic stability derivatives: p_1, p_2, p_3, p_4

Aerodynamic control derivatives: p_5, p_6

1

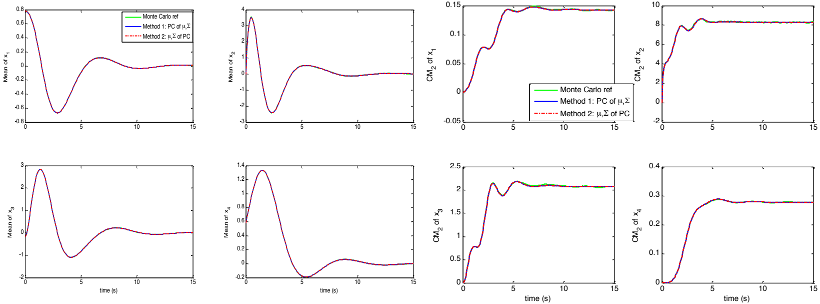
- $u_w \sim N(0,18)$, wind disturbance
- Assume uncertain stability derivatives $\sim U(\mathbf{p}_a,\mathbf{p}_b)$

	Lower bound	Upper bound
p_1	-0.0488	-0.0026
p_2	0.0013	0.0247
p_3	0.126	2.394
p_4	-3.3535	-0.1765



Evolution of nominal states

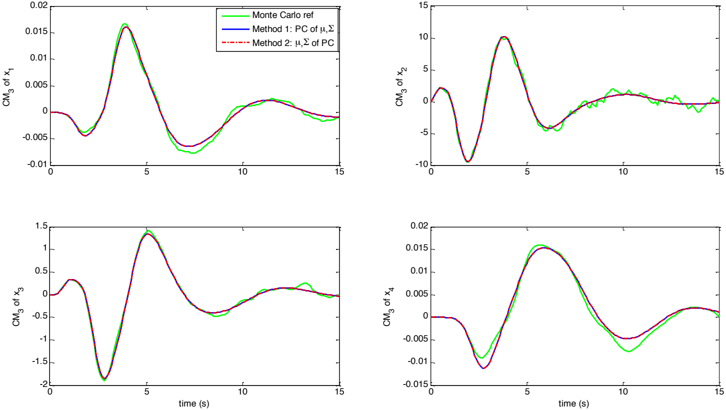
- $p_5 = 0.086$, $p_6 = -7.408$
- $g = 0.322$
- $K = [1.9890\ -0.2560\ -0.7589\ 1.0000]$
- $x_0 = [0.7929\ -0.0466\ -0.1871\ 0.5780]^T$

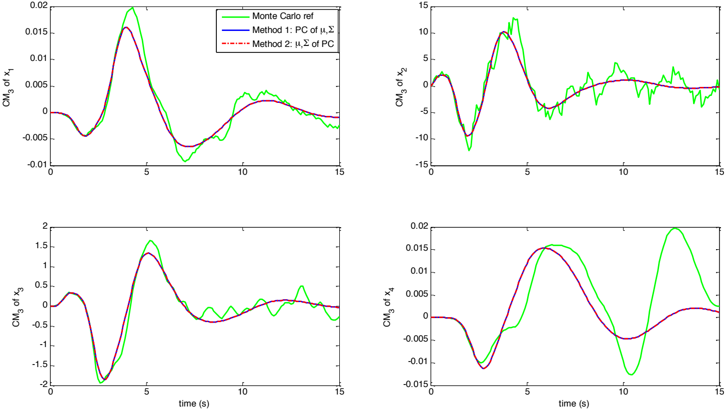


- Mean and variance of the states
- Monte Carlo runs = 100000
- PC order = 5, Legendre polynomials used as basis functions
- Runtime: >1h for Monte Carlo and less than 5s for method 1 and 1min for method 2.

Uncertainty Marriage

Hovering Helicopter: First Two Moments

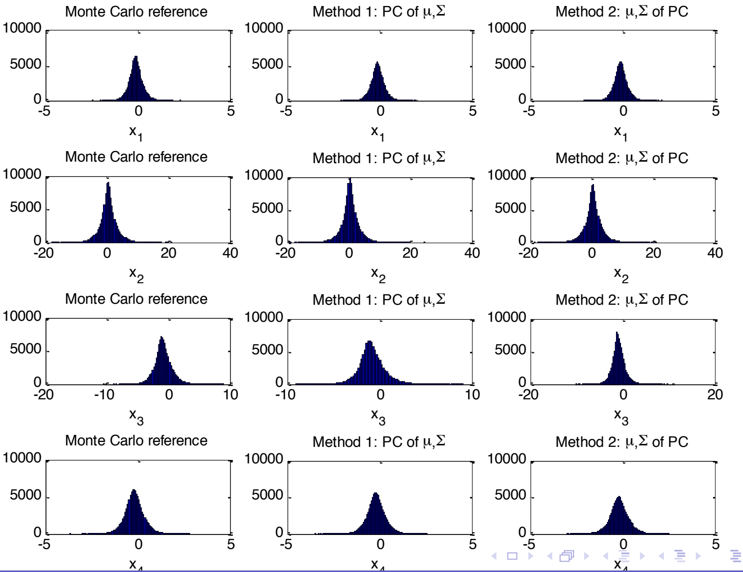




Monte Carlo runs = 10000

5

HOVERING HELICOPTER: HISTOGRAMS



State and Parameter Estimation

- Bayes' Theorem:

$$p(\Theta|\mathbf{y}_k) = \frac{p(\Theta)p(\mathbf{y}_k|\Theta)}{p(\mathbf{y}_k)} \tag{3}$$

where,

$$p(\mathbf{y}_k) = \int_\Theta p(\mathbf{y}_k|\Theta)p(\Theta)d\Theta = \mathscr{E}_\Theta\{p(\mathbf{y}_k|\Theta)\} \tag{4}$$

and $\mathbf{y}_k \in \mathbb{R}^b$ is the measurement data, provided by observation model

$$\mathbf{y}_k \triangleq \mathbf{y}(t_k) = \mathbf{h}(\mathbf{x}_k, \Theta, \nu_k), \quad \nu_k \sim p(\nu_k) \tag{5}$$

- Posterior statistics of parameter Θ:

$$\hat{\Theta}^+ = \mathscr{E}_\Theta\{\Theta\} = \frac{\int_\Theta \Theta p(\Theta)p(\mathbf{y}_k|\Theta)d\Theta}{\mathscr{E}_\Theta\{p(\mathbf{y}_k|\Theta)\}} = \frac{\mathscr{E}_\Theta\{\Theta p(\mathbf{y}_k|\Theta)\}}{\mathscr{E}_\Theta\{p(\mathbf{y}_k|\Theta)\}} \tag{6}$$

$$\mathbf{P}^+ = \int_\Theta \Theta\Theta^T p(\Theta|\mathbf{y}_k)d\Theta = \frac{\mathscr{E}_\Theta\{\Theta\Theta^T p(\mathbf{y}_k|\Theta)\}}{\mathscr{E}_\Theta\{p(\mathbf{y}_k|\Theta)\}} \tag{7}$$

- Higher order statistics:

$$\mathscr{E}^{+}\{\phi(\Theta)\} = \int_{\Theta} \phi(\Theta) p(\Theta|\mathbf{y}_k) d\Theta = \frac{\mathscr{E}_{\Theta}\{\phi(\Theta) p(\mathbf{y}_k|\Theta)\}}{\mathscr{E}_{\Theta}\{p(\mathbf{y}_k|\Theta)\}} \quad (8)$$

where,

$$\phi(\Theta) = \prod_{i=1}^{m} \theta_i^{n_i}, \quad n_i \geq 0 \quad (9)$$

- CUT quadrature scheme is used for evaluation of integrals.
- Now, posterior gPC coefficients can be found by matching posterior moments.

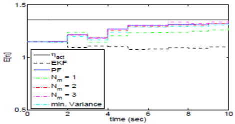
$$\ddot{x} + \eta\dot{x} + \alpha x + \beta x^3 = \sin(3t)$$

where, η ϵ U(0.9 , 1.4) , α ϵ U(-1.45 , -0.95), β=2

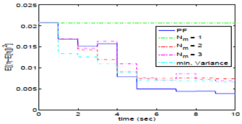
$$x(0) = -1, \qquad \dot{x}(0) = -1$$

$$\tilde{\mathbf{y}}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \mu(t)$$

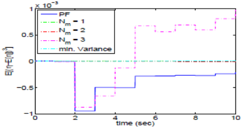
$$\mu(t) \simeq N(0, R) \qquad R = 0.05^2 I_{2\times 2}$$



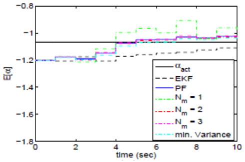
Mean



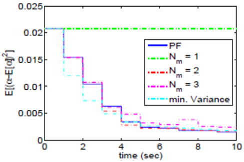
Variance



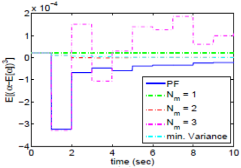
3rd Central Moment



Mean



Variance



3rd Central Moment

2

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[2] R. Madankan, P. Singla, T. Singh, and P. D. Scott, "Polynomial-chaos-based Bayesian approach for state and parameter estimations," AIAA Journal of Guidance, Control, and Dynamics (2013).

RMSE error between PF, min. Variance estimator, EKF, and gPC-Bayes method in estimation of different order of central moments of parameter η

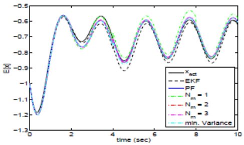
N_m	e_{m_1}	e_{m_2}	e_{m_3}
1	1.6894e+000	3.9390e-001	1.3371e-002
2	2.1844e-001	7.8388e-002	1.3233e-002
3	3.2892e-001	7.4929e-002	2.1433e-002
min. Variance	5.8136e-001	8.3721e-002	1.3263e-002
EKF	5.4728e+000	2.6102e-001	1.3245e-002

RMSE error between PF, min. Variance estimator, EKF, and gPC-Bayes method in estimation of different order of central moments of parameter α

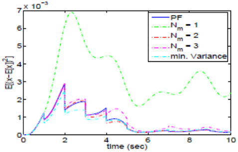
N_m	e_{m_1}	e_{m_2}	e_{m_3}
1	1.9049e+000	4.9394e-001	3.8996e-003
2	1.4937e-001	1.1782e-002	3.5675e-003
3	1.4523e-001	2.9626e-002	4.3245e-003
min. Variance	4.3090e-001	4.8743e-002	3.5880e-003
EKF	2.5449e+000	1.8308e-001	3.4666e-003

State and Parameter Estimation

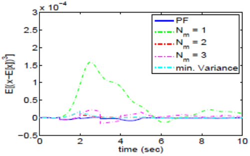
Duffing Oscillator



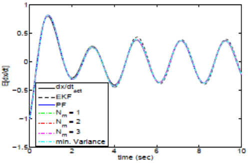
Mean



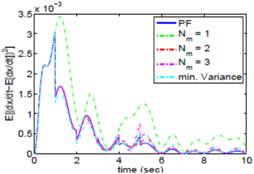
Variance



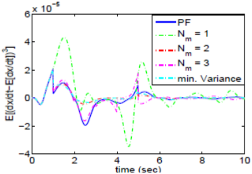
3^{rd} Central Moment



Mean



Variance



3^{rd} Central Moment

RMSE error between PF, min. Variance estimator, EKF, and gPC-Bayes method in estimation of different order of central moments of x(t)

N_m	e_{m_1}	e_{m_2}	e_{m_3}
1	7.0649e-001	8.5432e-002	1.9670e-003
2	6.7726e-002	2.3682e-003	1.5666e-004
3	6.1157e-002	6.2015e-003	3.0797e-004
min. Variance	1.7341e-001	5.7130e-003	1.7355e-004
EKF	9.5327e-001	1.6320e-002	9.2697e-005

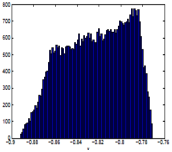
RMSE error between PF, min. Variance estimator, EKF, and gPC-Bayes method in estimation of different order of central moments of dx/dt

N_m	e_{m_1}	e_{m_2}	e_{m_3}
1	2.8121e-001	1.9142e-002	3.8337e-004
2	3.7524e-002	1.4365e-003	1.0563e-004
3	3.1312e-002	2.0126e-003	7.7402e-005
min. Variance	1.0039e-001	2.6557e-003	1.1128e-004
EKF	7.1761e-001	1.6451e-002	1.7959e-004

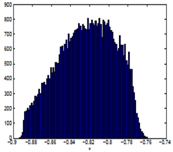
Computational Time for different estimation approaches

N_m	EKF	PF	min. Variance	gPC-Bayes
1	0.955887	3.6260e+004	2.1469e+001	2.0219e+002
2				2.8672e+002
3				1.6657e+004

- Error Analysis



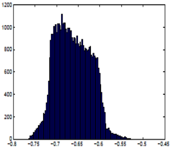
gPC-Bayes distribution of x after the update at t=1



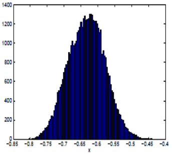
PF distribution of x after the update at t=1

method	e_{m_1}	e_{m_2}	e_{m_3}
PF	-8.2328e-001	8.5817e-004	-5.0996e-006
gPC-Bayes $(N_m = 3)$	-8.2322e-001	8.5724e-004	-5.0696e-006

First 3 central moments of x after the first update at t=1 sec. by using PF and gPC-Bayes method



gPC-Bayes distribution of x before the update at t=2



PF distribution of x before the update at t=2

method	e_{m_1}	e_{m_2}	e_{m_3}
PF	-6.3336e-001	2.9261e-003	1.7150e-006
gPC-Bayes $(N_m = 3)$	-6.6187e-001	1.5869e-003	1.0685e-006

First 3 central moments of x before the second update at t=2 sec. by using PF and gPC-Bayes method