Quadrature Methods

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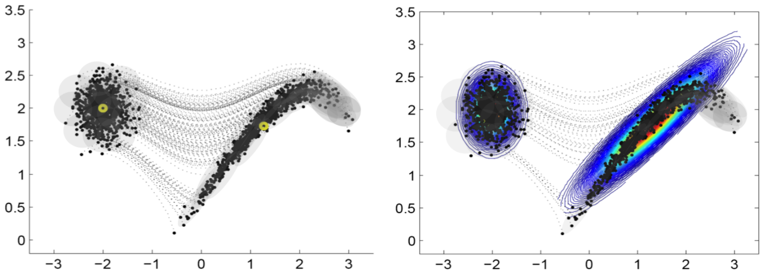
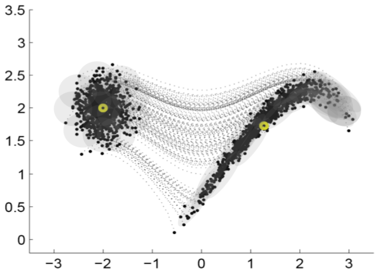


(a) (b)

- Different Approaches: Kolmogorov Equation, Monte Carlo Methods, generalized Polynomial Chaos (gPC).
- All these methods involves the evaluation of expectation integrals:

$$E[f(x)] = \int\int\cdots\int p(\mathbf{x})f(\mathbf{x})d\mathbf{x} \approx \sum_{i=1}^{n} w_i f(\mathbf{x}_i)$$



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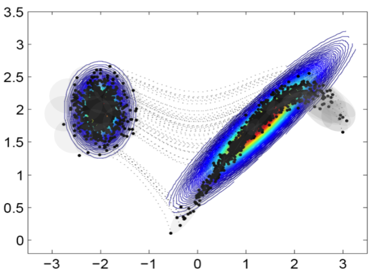

(d)

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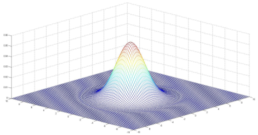
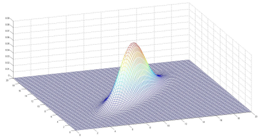
INTRODUCTION

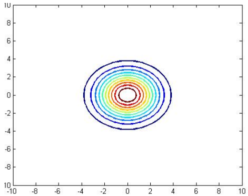
GAUSSIAN PDF UNDER LINEAR TRANSFORMATION



(e) pdf of x



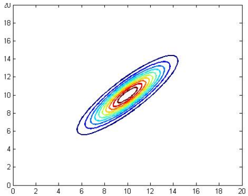
(a) pdf of y



(f) contours

LINEAR TRANSFORMATION

$$y = Ax + B$$



(b) contours

Figure: *Initial Gaussian pdf*

Figure: *Final Gaussian pdf*

ANALYTICAL MEAN AND COVARIANCE

$$E[y] = AE[x] + B \qquad E[(y - E[y])(y - E[y])^T] = \Sigma_y = A\Sigma_x A^T$$

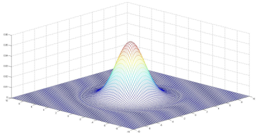


Figure: *Initial Gaussian pdf*

NONLINEAR TRANSFORMATION

$$y = f(x)$$

- Indirect information of the pdf
- No information about the shape of the pdf
- Can be easily evaluated by numerical integration

1. Computation of the pdf can be difficult.
2. Histogram can give some information about the shape.
3. Indirect information of the pdf: Statistics/Moments.

MOMENTS OF THE TRANSFORMED VARIABLE: FOR EXAMPLE IN THE 2D CASE

$$E[\mathbf{y}] = E[f(\mathbf{x})] = \int f(\mathbf{x})p(\mathbf{x})\,d\mathbf{x}$$

$$E[y_1^{n_1} y_2^{n_2}] = \int f_1(\mathbf{x})^{n_1} f_2(\mathbf{x})^{n_2}\,d\mathbf{x}$$

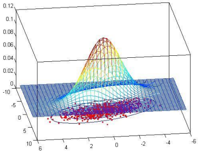
$$E[\mathbf{y}\mathbf{y}^T] = E[f(\mathbf{x})f(\mathbf{x})^T] = \int f(\mathbf{x})f(\mathbf{x})^T p(\mathbf{x})\,d\mathbf{x}$$

Dynamic System with uncertainty

$$\dot{x} = f(x, \rho) \quad \rho \sim p(\rho)$$

Solution for 1 set of parameters

$$x(t) = g(t, \rho^{(i)})$$



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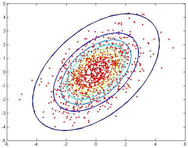
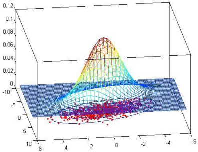


Uncertainty pdf and contours of $p(\rho)$



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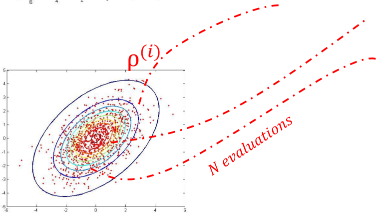

N points
Uncertainty pdf and contours of $p(\rho)$

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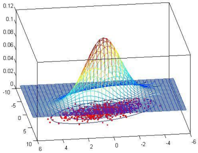

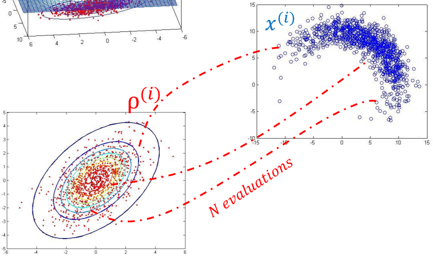

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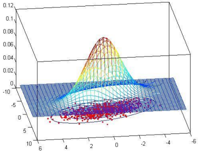
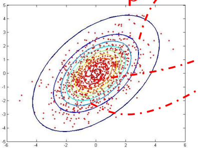
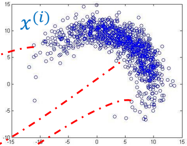




$\rho^{(i)}$

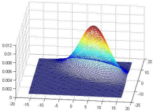
N evaluations

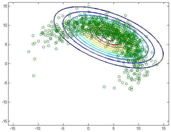


$x^{(i)}$

N points

Uncertainty pdf and contours of $p(\rho)$





Gaussian Approximation

$$\mu = \frac{1}{N}\sum_{i=1 \text{ to } N} x^{(i)} \qquad \Sigma = \frac{1}{N}\sum_{i=1 \text{ to } N} (x^{(i)}-\mu)\,(x^{(i)}-\mu)^T$$

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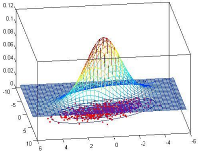

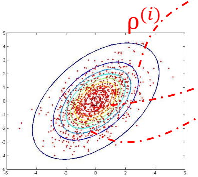

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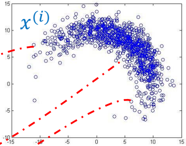

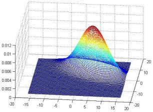

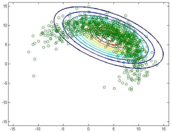

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Moment integral Approximation

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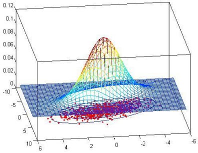

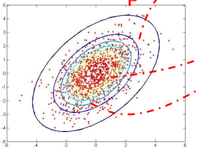

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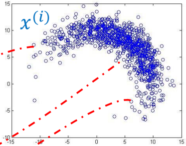
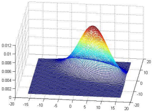
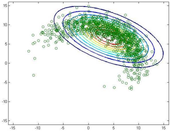


$x^{(i)}$

N evaluations





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$$E[x^n] = E[g(\rho)^n] = \int g(\rho)^n\, p(\rho)d\rho = \frac{1}{N}\sum g(\rho^{(i)})^n$$

More information with higher order moments

If direct shape is not important and only objective is to evaluate the moments, Monte Carlo runs can be avoided.

Objective

$$E[f(x)] = \int\int\cdots\int p(\mathbf{x})f(\mathbf{x})d\mathbf{x} \approx \sum_{i=1}^{N} w_i f(\mathbf{x}_i)$$

Particularly, we are interested in

Gaussian pdf $\Omega = [-\infty, \infty]$

$$p(\mathbf{x}) = exp(-\frac{1}{2}\sum_{i=1}^{n} x_i^2)$$

- All the points are real.
- All the weights are real and positive

Uniform pdf $\Omega = [-1, 1]$

$$p(\mathbf{x}) = \frac{1}{2^n}$$

- The weights add to 1
- *All the points lie within the support of the pdf*

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- *All the points lie within the support of the pdf*

Monte Carlo Methods

- Samples are drawn from the pdf $\mathbf{x}_i \sim p(\mathbf{x})$

The integral is approximated as an average of function evaluations at these samples

$$E[f(\mathbf{x})] = \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i)$$

- The error $\propto \frac{1}{\sqrt{N}}$, where N is the number of samples.
- This is considered to be a slow in convergence.
- Hence often a large number of samples are required to get sufficient accuracy.

Gaussian Quadratures

$$E[f(\mathbf{x})] = \sum_{i=1}^{N} w_i f(\mathbf{x}_i)$$

- The weights w_i and points $\mathbf{x}_i$ are deterministic.
- *1-D integrals:* N quadrature points are required to reproduce the expectation integrals of a polynomial with degree $\leq 2N-1$
- n-D integrals: Tensor product leads to exponential growth of points (N^n).

Gaussian pdf	*Hermite Polynomials*	*–Gauss-Hermite Quadrature*
Uniform pdf	*Legendre Polynomials*	*–Gauss-Legendre Quadrature*

Expectation Integral

$$E[f(\mathbf{x})] = \int_\Omega f(\mathbf{x})p(\mathbf{x})\,dx = \int\int\cdots\int f(\mathbf{x})p(\mathbf{x})dx_1dx_2\cdots dx_n$$

EXPECTATION INTEGRAL

$$E[f(\mathbf{x})] = \int_\Omega f(\mathbf{x})p(\mathbf{x})\,dx = \int\int\cdots\int f(\mathbf{x})p(\mathbf{x})dx_1dx_2\cdots dx_n$$

TAYLOR SERIES

$$E[f(\mathbf{x})] \approx \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_n=0}^{\infty}\frac{E[x_1^{N_1}x_2^{N_2}\cdots x_n^{N_n}]}{N_1!N_2!\cdots N_n!}\frac{\partial^{N_1+N_2+\cdots+N_n}f}{\partial x_1^{N_1}\partial x_2^{N_2}\cdots\partial x_n^{N_n}}(\mathbf{0})$$

- The problem of evaluating the expected value of $f(\mathbf{x})$ has reduced to computing higher order moments of the pdf.
- Increasing the number of terms in the Taylor series expansion $\Rightarrow$ Accurate evaluation of the expectation integral.

TAYLOR SERIES

$$E[f(\mathbf{x})] \approx \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_n=0}^{\infty} \frac{E[x_1^{N_1}x_2^{N_2}\cdots x_n^{N_n}]}{N_1!N_2!\cdots N_n!}\frac{\partial^{N_1+N_2+\cdots+N_n}f}{\partial x_1^{N_1}\partial x_2^{N_2}\cdots\partial x_n^{N_n}}(\mathbf{0})$$

WEIGHTED APPROXIMATION

$$E[f(\mathbf{x})] \approx \sum_{i=1}^{N} w_i f(\mathbf{x_i}) \approx \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\cdots\sum_{N_n=0}^{\infty} \frac{(\sum_{i=1}^{N} w_i\{x_{(i,1)}{}^{N_1}x_{(i,2)}{}^{N_2}\cdots x_{(i,n)}{}^{N_n}\})}{N_1!N_2!\cdots N_n!}\frac{\partial^{N_1+N_2+\cdots+N_n}f}{\partial x_1^{N_1}\partial x_2^{N_2}\cdots\partial x_n^{N_n}}(\mathbf{0})$$

COMPARING COEFFICIENTS OF $f(0)$ & ITS DERIVATIVES

$$\sum_{i=1}^{N} w_i\{x_{(i,1)}{}^{N_1}x_{(i,2)}{}^{N_2}\cdots x_{(i,n)}{}^{N_n}\} = E[x_1^{N_1}x_2^{N_2}\cdots x_n^{N_n}]$$

We just need to match moments in input space, $\mathbf{x}$

Expectation Integral

$$E[f(x)] = \int_{\Omega} f(x)p(x)\,dx$$

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$$E[f(x)] = \int_{\Omega} f(x)p(x)\,dx$$

If $p(x) = \mathcal{N}(x:0,1)$ and $\Omega = (-\infty, \infty)$, the moments of the Gaussian pdf are

$$E[1] = 1, \quad E[x] = 0, \quad E[x^2] = 1, \quad E[x^3] = 0, \quad E[x^4] = 3$$

So, if $f(x) = P_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$$\int_{\Omega} f(x)p(x)\,dx = \int_{\Omega} P_3(x)\mathcal{N}(x:0,1)\,dx$$

$$= \int_{\Omega} a_3x^3\mathcal{N}(x:0,1)\,dx + \int_{\Omega} a_2x^2\mathcal{N}(x:0,1)\,dx + \int_{\Omega} a_1x\mathcal{N}(x:0,1)\,dx + \int_{\Omega} a_0\mathcal{N}(x:0,1)\,dx$$

$$= a_3E[x^3] + a_2E[x^2] + a_1E[x] + a_0E[1] \quad (\textit{Integrating Polynomials} \Rightarrow \textit{Capturing moments})$$

$$= a_0 + a_2$$

EXPECTATION INTEGRAL

$$E[f(x)] = \int_\Omega f(x)p(x)\,dx$$

If $p(x) = \frac{1}{2}$ and $\Omega = [-1,1]$, the moments of the Uniform pdf are:

$$E[1] = 1, \quad E[x] = 0, \quad E[x^2] = \frac{1}{3}, \quad E[x^3] = 0, \quad E[x^4] = \frac{1}{5}$$

So, if $f(x) = P_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$$\int_\Omega f(x)p(x)\,dx = \int_\Omega P_3(x)\frac{1}{2}\,dx$$

$$= \int_\Omega a_3x^3\frac{1}{2}\,dx + \int_\Omega a_2x^2\frac{1}{2}\,dx + \int_\Omega a_1x\frac{1}{2}\,dx + \int_\Omega a_0\frac{1}{2}\,dx$$

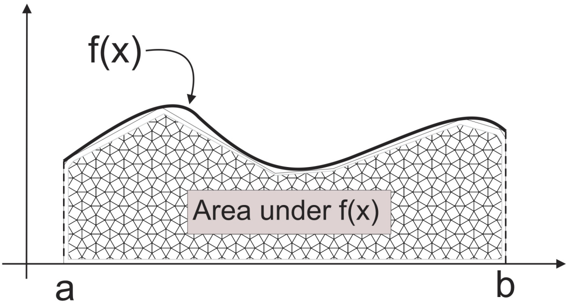
$$= \frac{a_3}{2}E[x^3] + \frac{a_2}{2}E[x^2] + \frac{a_1}{2}E[x] + \frac{a_0}{2}E[1] \quad \textit{(Integrating Polynomials} \Rightarrow \textit{Capturing moments)}$$

$$= \frac{a_0}{2} + \frac{a_2}{6}$$

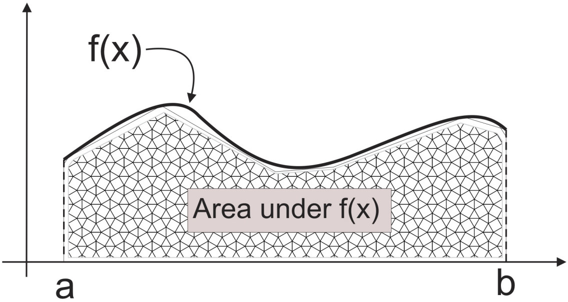
Expectation Integral

$$E[f(x)] = \int_{\Omega} f(x)p(x)\,dx$$

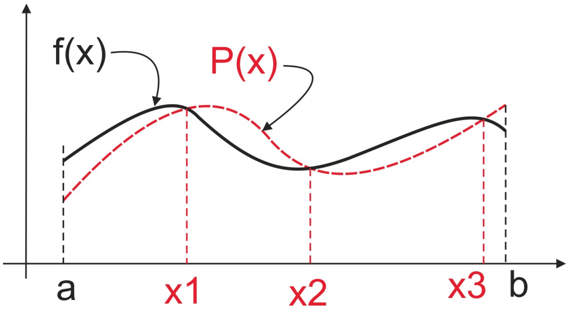
i.e. it is required to find the area under the curve $f(x)$



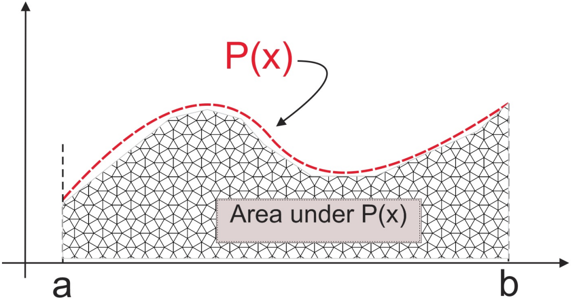
Given the function $f(x)$



Approximate $f(x)$ by a polynomial $P(x)$



Integrate the polynomial $P(x)$



For example consider the 3rd order polynomial approximation:

$$P_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

4 variables $\Rightarrow$ 4 equations are needed. Consider 2 points x_1 and x_2

$$\begin{aligned}
f(x_1) =& P_3(x_1) = a_3x_1^3 + a_2x_1^2 + a_1x_1 + a_0 \\
f(x_2) =& P_3(x_2) = a_3x_2^3 + a_2x_2^2 + a_1x_2 + a_0 \\
f'(x_1) =& P_3'(x_1) = 3a_3x_1^2 + 2a_2x_1 + a_1 \\
f'(x_2) =& P_3'(x_2) = 3a_3x_2^2 + 2a_2x_2 + a_1
\end{aligned}$$

Solve for a_0, a_1, a_2, a_3

$$\begin{aligned}
a_0 &= \frac{f(x_2)x_1^2(x_1 - 3x_2) + x_2\left(f(x_1)(3x_1 - x_2)x_2 - x_1(x_1 - x_2)\left(x_2 f'(x_1) + x_1 f'(x_2)\right)\right)}{(x_1 - x_2)^3} \\
a_1 &= \frac{-6f(x_1)x_1x_2 + 6f(x_2)x_1x_2 + (x_1 - x_2)\left(x_2^2 f'(x_1) + x_1^2 f'(x_2) + 2x_1x_2\left(f'(x_1) + f'(x_2)\right)\right)}{(x_1 - x_2)^3} \\
a_2 &= \frac{3f(x_1)(x_1 + x_2) - 3f(x_2)(x_1 + x_2) - (x_1 - x_2)\left(x_2\left(2f'(x_1) + f'(x_2)\right) + x_1\left(f'(x_1) + 2f'(x_2)\right)\right)}{(x_1 - x_2)^3} \\
a_3 &= \frac{-2f(x_1) + 2f(x_2) + (x_1 - x_2)\left(f'(x_1) + f'(x_2)\right)}{(x_1 - x_2)^3}
\end{aligned}$$

For Gaussian pdf

$$\int_{\Omega} f(x)\,dx \approx \int_{\Omega} P_3(x)\,dx = a_2 + a_0$$

Force the coefficients of $f'(x_1)$ and $f'(x_2)$ to 0

$$\frac{2x_2 + x_1(1+x_2^2)}{(x_1-x_2)^2} = 0 \qquad \frac{2x_1 + x_2 + x_1^2 x_2}{(x_1-x_2)^2} = 0$$

$$x_1 = -1 \qquad x_2 = 1$$

$$w_1 = \frac{1}{2} \qquad w_2 = \frac{1}{2}$$

x_1 and x_2 are exactly the roots of the 2^{nd} order Hermite Polynomial $x^2 - 1$

For Uniform pdf

$$\int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} P_3(x)\,dx = 2a_0 + \frac{2a_2}{3}$$

Force the coefficients of $f'(x_1)$ and $f'(x_2)$ to 0

$$\frac{2x_2 + x_1(1+3x_2^2)}{3(x_1-x_2)^2} = 0 \qquad \frac{2x_1 + x_2 + 3x_1^2 x_2}{3(x_1-x_2)^2} = 0$$

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$$= \frac{(3x_2^2 - x_2^3 + 3x_1(1+x_2^2))}{(x_1-x_2)^3} f(x_1) + \frac{(-3x_1 + x_1^3 - 3x_2 - 3x_1^2 x_2)}{(x_1-x_2)^3} f(x_2)$$

$$- \frac{2x_2 + x_1(1+x_2^2)}{(x_1-x_2)^2} f'(x_1) - \frac{2x_1 + x_2 + x_1^2 x_2}{(x_1-x_2)^2} f'(x_2)$$

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Expectation Integral

Polynomial Approximation

For Gaussian pdf

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Hence, for Gaussian quadratures, one only needs

- 2 points to integrate all polynomials of degree 3 or less

The procedure can be repeated using a 5^{th} order polynomial with 6 coefficients. One can use 3 function evaluations and 3 function derivative evaluations at 3 points to form 6 equations. This way one has

- 3 points to integrate all polynomials of degree 5 or less

In general Gaussian Quadratures only need

- N points to integrate all polynomials of degree $2N-1$ or less

$$\begin{array}{lll} p(x) \sim Gaussian & Hermite\ Polynomial & Gauss-Hermite\ Quadrature \\ p(x) \sim Uniform & Legendre\ Polynomial & Gauss-Legendre\ Quadrature \end{array}$$

- Alternatively, one can use Nested Quadrature such as Clenshaw-Curtis[1] Quadrature
- Higher order points make use of lower order points (*hence nested*)
- Clenshaw curtis needs $N-1$ points to integrate polynomials of degree N or less
- Gaussian Quadratures need N points to integrate polynomials of degree $2N-1$ or less

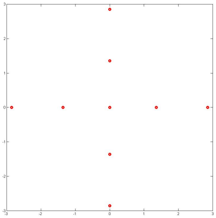

Figure: *1D points*

Higher dimension Gaussian Quadrature are constructed from Tensor product of 1D points

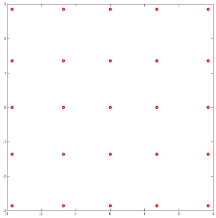

Figure: *2D points*

[1] Trefethen, Lloyd N. "Is Gauss quadrature better than Clenshaw-Curtis?." SIAM review 50.1 (2008): 67-87.

$$Q_k^{(n)} = \sum \left((Q_i^{(1)} - Q_{i-1}^{(1)}) \otimes Q_{k-i+1}^{(n-1)} \right) [f]$$

- $Q_k^{(n)}$ is the sparse grid rule of dimension n and of $2k-1$ degree exactness.
- where $Q_i^{(1)}$ is the 1D Gaussian Quadrature rule with i points.
- The Smolyak sparse grid scheme[1] has fewer points than the Gaussian Quadrature Product rule.

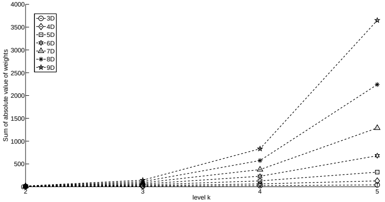

Figure: *Sum of absolute value of the weights*

- This algorithm can contain negative weights.
- If $\sum |w_i| >> 1$, then the quadrature rule introduces large roundoff errors[2].

[1] T. Gerstner and M. Griebel. “Numerical integration using sparse grids.” Numerical algorithms 18.3-4 (1998): 209-232.
[2] A. H. Stroud, “Approximate Calculation of Multiple Integrals,” Prentice Hall, 1971.

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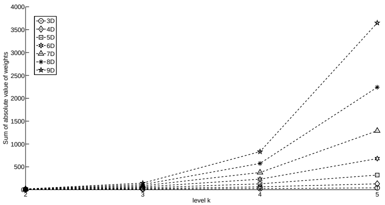

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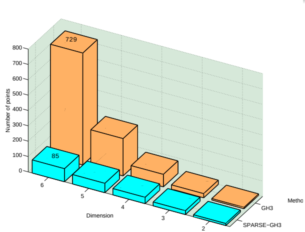
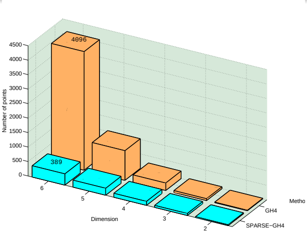
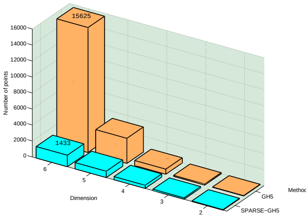


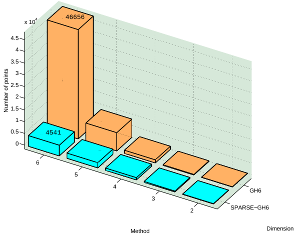
(a) Polynomial exactness 5 or less



(b) Polynomial exactness 7 or less



(c) Polynomial exactness 9 or less



(d) Polynomial exactness 11 or less

- Developed specifically for the Gaussian pdf.
- 3^{rd} order rule with $2n+1$ points.
- Can integrate all odd degree monomials due to symmetry of the points chosen.
- The tuning parameter κ is selected such that $n+\kappa=3$. This makes the central weight negative after dimension 3.
- The points are constrained to be on the orthogonal axes (or Principal axes).

UNSCENTED TRANSFORM

$$X_0 = (0,\cdots,0) \qquad W_0 = \kappa/(n+\kappa)$$
$$X_i = \sqrt{(n+\kappa)}I_i \qquad W_i = 1/[2(n+\kappa)]$$
$$X_{i+n} = -\sqrt{(n+\kappa)}I_i \qquad W_{i+n} = 1/[2(n+\kappa)]$$

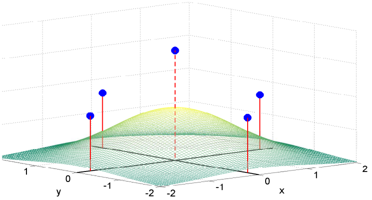


Figure: *Selection of points in 2D*

S. Julier, J. Uhlmann, H. F. Durrant-Whyte, "A new method for the nonlinear transformation of means and covariances in filters and estimators," Automatic Control, IEEE Transactions on , vol.45, no.3, March 2000.

Assuming i.i.d Gaussian random variables $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$ with zero mean and identity covariance, the higher order moments up to fourth order are given as:

MOMENTS OF A GAUSSIAN PDF

$$E[x_i^2] = 1, \quad E[x_i x_j] = 0, \quad E[x_i^4] = 3$$
$$E[x_i^3 x_j] = 0, \quad E[x_i^2 x_j^2] = 1, \quad E[x_i^2 x_j x_k] = 0$$
$$E[x_i x_j x_k x_l] = 0,$$

The central point on the origin (mean) has weight w_0. When enumerated the set of points are $\{(r_1, 0, \cdots, 0), (0, r_1, \cdots, 0), (0, 0, \cdots, r_1)\ (-r_1, 0, \cdots, 0), (0, -r_1, \cdots, 0), (0, 0, \cdots, -r_1)\ , (0, 0, \cdots, 0)\}$ and all the points have the same weight w_1. The corresponding MCE upto 4^{th} order are:

$$E[x_i^0] \equiv w_0 + 2nw_1 = 1 \qquad E[x_i^4] \equiv 2r_1^4 w_1 = 3$$
$$E[x_i^2] \equiv 2r_1^2 w_1 = 1 \qquad E[x_i^2 x_j^2] \equiv 0 \neq 1$$

The cross order moment cannot be satisfied with this choice of points (as all points have only one non-zero coordinate)

$$E[x_i^0] \equiv w_0 + 2nw_1 = 1 \qquad E[x_i^4] \equiv 2r_1^4 w_1 = 3$$
$$E[x_i^2] \equiv 2r_1^2 w_1 = 1$$

From the 3rd equation, w_1 can be solved as:

$$w_1 = \frac{1}{2r_1^2} \quad \Rightarrow r_1 = \sqrt{3}$$
$$\Rightarrow w_1 = \frac{1}{6}$$
$$\Rightarrow w_0 = 1 - \frac{n}{3}$$

Negative central weight after dimension 3

Alternatively, choose $r_1 = \sqrt{n+\kappa}$, where κ is a parameter and n is dimension

$$w_1 = \frac{1}{2(n+\kappa)}$$
$$w_0 = 1 - 2nw_1 = 1 - \frac{n}{n+\kappa} = \frac{\kappa}{n+\kappa}$$
$$2r_1^2 w_1 = 2(n+\kappa)^2 \frac{1}{2(n+\kappa)} = n+\kappa = 3$$

$n+\kappa = 3$ to capture only one 4^{th} order moment, however, central weight becomes negative for $n > 3$.

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$n+\kappa = 3$ to capture only one 4^{th} order moment, however, central weight becomes negative for $n > 3$.

- For zero mean and identity covariance Gaussian pdf: the points on the perpendicular cartesian axes can be represented as
- $(\pm r_1, 0, \cdots, 0)$, $(0, \pm r_1, \cdots, 0) \cdots$, $(0, 0, \cdots, \pm r_1)$ and $(0, 0, \cdots, 0)$.
- Any cross order moment such as $E[x_1^2 x_2^2]$ becomes zero by such selection of points.
- One would have to *'look for additional axes that are able to capture higher order moments'*

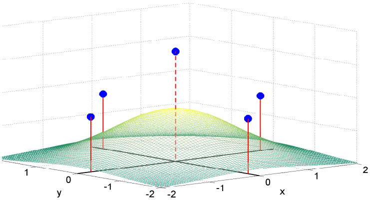


Figure: *Unscented Transform for 2D*

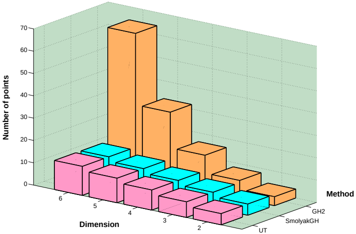


Figure: *Comparison of 3^{rd} degree rules*

Moment Constraint Equations(*MCEs*):

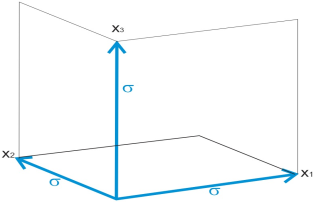
$$\sum_{i=1}^{N} w_i \{ {x_{(i,1)}}^{n_1} {x_{(i,2)}}^{n_2} \cdots {x_{(i,n)}}^{n_n} \} = E[x_1^{n_1} x_2^{n_2} \cdots x_N^{n_n}]$$

A Fully Symmetric Set

A set of points is called fully symmetric if it is closed under all coordinate and sign permutations.

- For example consider the set $\mathbf{X} = \{\mathbf{x_1}, \mathbf{x_2}, \cdots\}$ to be a fully symmetric set and if $\mathbf{x_i} = [a, b]^T \in \mathbf{X}$, then $\{[b,a]^T, [-a,b]^T, [a,-b]^T, [-a,-b]^T, [-b,a]^T, [b,-a]^T, [-b,-a]^T\} \in \mathbf{X}$.

σ: Represents the *Principal axes* (or orthogonal axis) in the cartesian coordinate space. Each point on the principal axis is denoted as σ_i



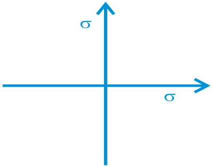
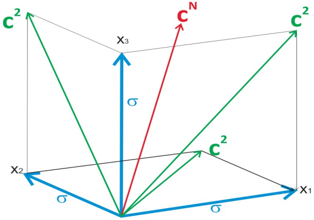


TABLE: Fully Symmetric set of points

Type	Sample Point	No. of points
σ	$(1,0,0,\cdots,0)$	$2n$
c^M	$(\underbrace{1,1,\cdots,1}_{M},\underbrace{0,0,\cdots,0}_{n-M})$	$2^M\binom{n}{M}$
$s^n(h)$	$(h,1,1,\cdots,1)$	$n2^n$

c^M: Represents the M^{th} *Conjugate axes (fully symmetric set)* and the corresponding points are enumerated as c_i^M



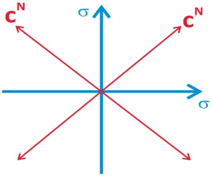


Table: Fully Symmetric set of points

Type	Sample Point	No. of points
σ	$(1,0,0,\cdots,0)$	$2n$
c^M	$(\underbrace{1,1,\cdots,1}_{M},\underbrace{0,0,\cdots,0}_{n-M})$	$2^M\binom{n}{M}$
$s^n(h)$	$(h,1,1,\cdots,1)$	$n2^n$

$s^n(h)$: Represents the *Scaled Conjugate axes* and the points are denoted as $s_i^N(h)$. The parameter h is a scaling factor.

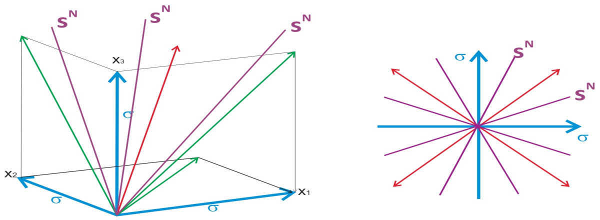
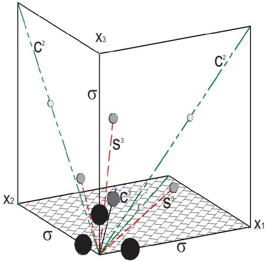
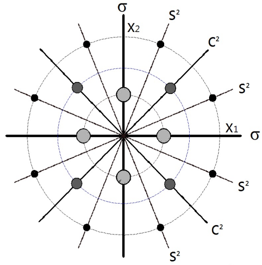


TABLE: Fully Symmetric set of points

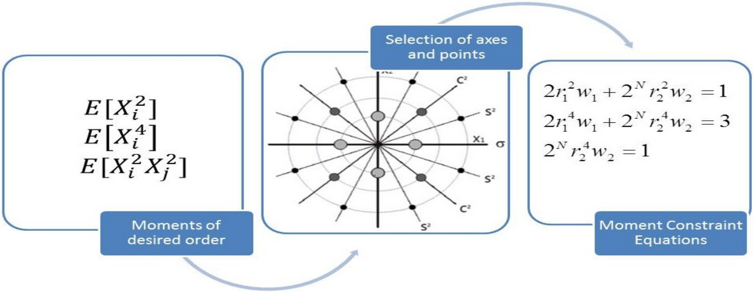
Type	Sample Point	No. of points
σ	$(1,0,0,\cdots,0)$	$2n$
c^M	$(\underbrace{1,1,\cdots,1}_{M},\underbrace{0,0,\cdots,0}_{n-M})$	$2^M\binom{n}{M}$
$s^n(h)$	$(h,1,1,\cdots,1)$	$n2^n$



X3
C²
C²
σ
S³
C³
S³
X2
X1
σ
σ



σ
X2
S²
C²
S²
X1
σ
S²
C²
S²



Selection of axes and points
$E[X_i^2]$
$E[X_i^4]$
$E[X_i^2 X_j^2]$
Moments of desired order
$2r_1^2w_1 + 2^N r_2^2 w_2 = 1$
$2r_1^4w_1 + 2^N r_2^4 w_2 = 3$
$2^N r_2^4 w_2 = 1$
Moment Constraint Equations

Let $[x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k, \ldots, x_l, \ldots, x_n]$ be a n-dimensional random vector

Gaussian pdf

Moment	value	Moment	value	Moment	value
$E[x_i^2]$	1	$E[x_i^4]$	3	$E[x_i^2 x_j^2]$	1
$E[x_i^6]$	15	$E[x_i^4 x_j^2]$	3	$E[x_i^2 x_j^2 x_k^2]$	1
$E[x_i^8]$	105	$E[x_i^6 x_j^2]$	15	$E[x_i^4 x_j^4]$	9
$E[x_i^4 x_j^2 x_k^2]$	3	$E[x_i^2 x_j^2 x_k^2 x_l^2]$	1		

Uniform pdf

Moment	value	Moment	value		
$E[x_i^2]$	$\frac{1}{3}$	$E[x_i^4]$	$\frac{1}{5}$	$E[x_i^2 x_j^2]$	$\frac{1}{9}$
$E[x_i^6]$	$\frac{1}{7}$	$E[x_i^4 x_j^2]$	$\frac{1}{15}$	$E[x_i^2 x_j^2 x_k^2]$	$\frac{1}{27}$
$E[x_i^8]$	$\frac{1}{9}$	$E[x_i^6 x_j^2]$	$\frac{1}{21}$	$E[x_i^4 x_j^4]$	$\frac{1}{25}$
$E[x_i^4 x_j^2 x_k^2]$	$\frac{1}{45}$	$E[x_i^2 x_j^2 x_k^2 x_l^2]$	$\frac{1}{81}$		

TABLE: Fully Symmetric set of points for CUT4

Dimension	σ	c^2	c^3	c^4	c^5	c^6	c^7	c^8	c^9	s^n
2	4	4	-	-	-	-	-	-	-	8
3	6	12	8	-	-	-	-	-	-	24
4	8	24	32	16	-	-	-	-	-	64
5	10	40	80	80	32	-	-	-	-	160
6	12	60	160	240	192	64	-	-	-	384
7	14	84	280	560	672	448	128	-	-	896
8	16	112	448	1120	1792	1792	1024	256	-	2048
9	18	144	672	2016	4032	5376	4608	2304	512	4608

Choosing points as

- σ : $(\pm r_1, 0, 0, \ldots, 0)$ with weight w_1
- c^2 : $(\pm r_2, \pm r_2, 0, \ldots, 0)$ with weight w_2

$$\begin{aligned} E[x_i^2] &\equiv 2r_1^2 w_1 + 4(n-1) r_2^2 w_2 = 1 \\ E[x_i^4] &\equiv 2r_1^4 w_1 + 4(n-1) r_2^4 w_2 = 3 \\ E[x_i^2 x_j^2] &\equiv 4r_2^4 w_2 = 1 \\ E[x_i^0] &\equiv 1 - 2nw_1 - 2n(n-1) w_2 = w_0 \end{aligned}$$

$$w_2 = \frac{1}{4r_2^4} \quad , \quad w_1 = \frac{4-n}{2r_1^4}$$

$$r_1^2 r_2^2 = r_1^2(n-1) + r_2^2(4-n)$$

Table: Fully Symmetric set of points for CUT4

Dimension	σ	c^2	c^3	c^4	c^5	c^6	c^7	c^8	c^9	s^n
2	4	4	-	-	-	-	-	-	-	8
3	6	12	8	-	-	-	-	-	-	24
4	8	24	32	16	-	-	-	-	-	64
5	10	40	80	80	32	-	-	-	-	160
6	12	60	160	240	192	64	-	-	-	384
7	14	84	280	560	672	448	128	-	-	896
8	16	112	448	1120	1792	1792	1024	256	-	2048
9	18	144	672	2016	4032	5376	4608	2304	512	4608

Choosing points as

- $\sigma : (\pm r_1, 0, 0, \dots, 0)$ with weight w_1
- $c^2 : (\pm r_2, \pm r_2, 0, \dots, 0)$ with weight w_2

$$\begin{aligned} E[x_i^2] &\equiv 2r_1^2 w_1 + 4(n-1) r_2^2 w_2 = 1 \\ E[x_i^4] &\equiv 2r_1^4 w_1 + 4(n-1) r_2^4 w_2 = 3 \\ E[x_i^2 x_j^2] &\equiv 4 r_2^4 w_2 = 1 \\ E[x_i^0] &\equiv 1 - 2n w_1 - 2n(n-1) w_2 = w_0 \end{aligned}$$

$$w_2 = \frac{1}{4r_2^4} \quad , \quad w_1 = \frac{4-n}{2r_1^4}$$

$$r_1^2 r_2^2 = r_1^2(n-1) + r_2^2(4-n)$$

TABLE: Fully Symmetric set of points for CUT4[2]

Dimension	σ	c^2	c^3	c^4	c^5	c^6	c^7	c^8	c^9	s^n
2	4	4	-	-	-	-	-	-	-	8
3	6	12	8	-	-	-	-	-	-	24
4	8	24	32	16	-	-	-	-	-	64
5	10	40	80	80	32	-	-	-	-	160
6	12	60	160	240	192	64	-	-	-	384
7	14	84	280	560	672	448	128	-	-	896
8	16	112	448	1120	1792	1792	1024	256	-	2048
9	18	144	672	2016	4032	5376	4608	2304	512	4608

Choosing points as

- σ : $(\pm r_1, 0, 0, \ldots, 0)$ with weight w_1
- c^n : $(\pm r_2, \pm r_2, \ldots, \pm r_2)$ with weight w_2

$$
\begin{aligned}
E[x_i^2] &\equiv 2r_1^2 w_1 + 2^n r_2^2 w_2 = 1 \\
E[x_i^4] &\equiv 2r_1^4 w_1 + 2^n r_2^4 w_2 = 3 \\
E[x_i^2 x_j^2] &\equiv 2^n r_2^4 w_2 = 1 \\
E[x_i^0] &\equiv 1 - 2n w_1 - 2^n w_2 = w_0
\end{aligned}
$$

$$r_1 = \sqrt{\frac{n+2}{2}}, \qquad r_2 = \sqrt{\frac{n+2}{n-2}}$$

$$w_1 = \frac{1}{r_1^4} = \frac{4}{(n+2)^2}, \qquad w_2 = \frac{1}{2^n r_2^4} = \frac{(n-2)^2}{2^n (n+2)^2}$$

[2] N. Adhurthi, P. Singla and T. Singh, "The Conjugate Unscented Transform - An Approach to Evaluate Multi-Dimensional Expectation Integrals," *2012 American Control Conference, Montréal, Canada, June 27-June 29, 2012.*

Table: Fully Symmetric set of points for CUT6

Dimension	σ	c^2	c^3	c^4	c^5	c^6	c^7	c^8	c^9	s^n
2	4	4	-	-	-	-	-	-	-	8
3	6	12	8	-	-	-	-	-	-	24
4	8	24	32	16	-	-	-	-	-	64
5	10	40	80	80	32	-	-	-	-	160
6	12	60	160	240	192	64	-	-	-	384
7	14	84	280	560	672	448	128	-	-	896
8	16	112	448	1120	1792	1792	1024	256	-	2048
9	18	144	672	2016	4032	5376	4608	2304	512	4608

Choosing points as

- $\sigma : (\pm r_1, 0, 0, \ldots, 0)$ with weight w_1
- $c^n : (\pm r_2, \pm r_2, \ldots, \pm r_2)$ with weight w_2
- $c^2 : (\pm r_3, \pm r_3, 0, \ldots, 0)$ with weight w_3

- Numerical solvers for polynomial system of equations can be exploited to solve these equations.

$$
\begin{aligned}
2r_1^2 w_1 + 2^n r_2^2 w_2 + 4(n-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^n r_2^4 w_2 + 4(n-1) r_3^4 w_3 &= 3 \\
2^n r_2^4 w_2 + 4 r_3^4 w_3 &= 1 \\
2r_1^6 w_1 + 2^n r_2^6 w_2 + 4(n-1) r_3^6 w_3 &= 15 \\
2^n r_2^6 w_2 + 4 r_3^6 w_3 &= 3 \\
2^n r_2^6 w_2 &= 1 \\
1 - 2n w_1 - 2^n w_2 - 2n(n-1) w_3 &= w_0
\end{aligned}
$$

Table: Fully Symmetric set of points for CUT8

Dimension	σ	c^2	c^3	c^4	c^5	c^6	c^7	c^8	c^9	s^n
2	4	4	-	-	-	-	-	-	-	8
3	6	12	8	-	-	-	-	-	-	24
4	8	24	32	16	-	-	-	-	-	64
5	10	40	80	80	32	-	-	-	-	160
6	12	60	160	240	192	64	-	-	-	384

Choosing points as

- σ : $(\pm r_1, 0, 0, \ldots, 0)$ with weight w_1.
- c^n : $(\pm r_2, \pm r_2, \ldots, \pm r_2)$ with weight w_2.
- c^2 : $(\pm r_3, \pm r_3, 0, \ldots, 0)$ with weight w_3.
- c^n : $(\pm r_4, \pm r_4, \ldots, \pm r_4)$ with weight w_4.
- c^3 : $(\pm r_5, \pm r_5, \pm r_5, 0, \ldots, 0)$ with weight w_5.
- s^n : $(\pm h r_6, \pm r_6, \ldots, \pm r_6)$ with weight w_6.

- 11 equations and 13 variables.
- Tractable solution obtained by assuming values for h.
- Solved by Numerical polynomial system solvers such as BERTINI.
- Choose the real and positive solutions.

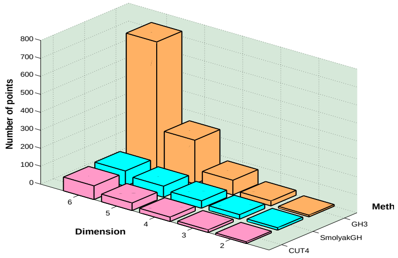
$$2r_1^2w_1 + 32r_2^2w_2 + 16r_3^2w_3 + 32r_4^2w_4 + 48r_5^2w_5 + 128r_6^2w_6 + 32h^2r_6^2w_6 = 1$$
$$2r_1^4w_1 + 32r_2^4w_2 + 16r_3^4w_3 + 32r_4^4w_4 + 48r_5^4w_5 + 128r_6^4w_6 + 32h^4r_6^4w_6 = 3$$
$$32r_2^4w_2 + 4r_3^4w_3 + 32r_4^4w_4 + 24r_5^4w_5 + 96r_6^4w_6 + 64h^2r_6^4w_6 = 1$$
$$2r_1^6w_1 + 32r_2^6w_2 + 16r_3^6w_3 + 32r_4^6w_4 + 48r_5^6w_5 + 128r_6^6w_6 + 32h^6r_6^6w_6 = 15$$
$$32r_2^6w_2 + 4r_3^6w_3 + 32r_4^6w_4 + 24r_5^6w_5 + 96r_6^6w_6 + 32h^2r_6^6w_6 + 32h^4r_6^6w_6 = 3$$
$$32r_2^6w_2 + 32r_4^6w_4 + 8r_5^6w_5 + 64r_6^6w_6 + 96h^2r_6^6w_6 = 1$$
$$2r_1^8w_1 + 32r_2^8w_2 + 16r_3^8w_3 + 32r_4^8w_4 + 48r_5^8w_5 + 128r_6^8w_6 + 32h^8r_6^8w_6 = 105$$
$$32r_2^8w_2 + 4r_3^8w_3 + 32r_4^8w_4 + 24r_5^8w_5 + 96r_6^8w_6 + 32h^2r_6^8w_6 + 32h^6r_6^8w_6 = 15$$
$$32r_2^8w_2 + 4r_3^8w_3 + 32r_4^8w_4 + 24r_5^8w_5 + 96r_6^8w_6 + 64h^4r_6^8w_6 = 9$$
$$32r_2^8w_2 + 32r_4^8w_4 + 8r_5^8w_5 + 64r_6^8w_6 + 64h^2r_6^8w_6 + 32h^4r_6^8w_6 = 3$$
$$32r_2^8w_2 + 32r_4^8w_4 + 32r_6^8w_6 + 128h^2r_6^8w_6 = 1$$

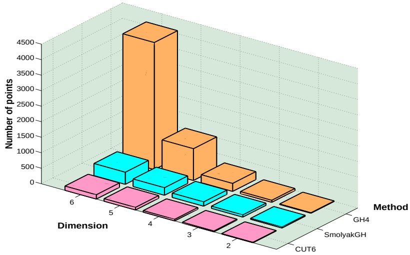
TABLE: CUT4

	Position	Weights
$1 \le i \le 2N$	$X_i = r_1 \sigma_i$	$W_i = w_1$
$1 \le i \le 2^N$	$X_{i+2N} = r_2 c_i^N$	$W_{i+2N} = w_2$
Central weight	$X_0 = \mathbf{0}$	$W_0 = w_0$
$n = 2N + 2^N \ (+1)$		

TABLE: CUT6, $(N \le 6)$

	Position	Weights
$1 \le i \le 2N$	$X_i = r_1 \sigma_i$	$W_i = w_1$
$1 \le i \le 2^N$	$X_{i+2N} = r_2 c_i^N$	$W_{i+2N} = w_2$
$1 \le i \le 2N(N-1)$	$X_{i+2N+2^N} = r_3 c_i^2$	$W_{i+2N+2^N} = w_3$
Central weight	$X_0 = \mathbf{0}$	$W_0 = w_0$
$n = 2N^2 + 2^N + 1$		



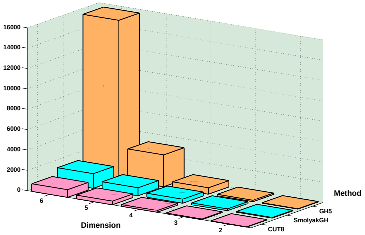


N. Adhurthi et al., "The Conjugate Unscented Transform - An Approach to Evaluate Multi-Dimensional Expectation Integrals," *2012 ACC*.

Table: Sigma Points for CUT8, $(2 \le N \le 6)$

	Position	Weights
$1 \le i \le 2N$	$X_i = r_1 \sigma_i$	$W_i = w_1$
$1 \le i \le 2^N$	$X_{i+2N} = r_2 c_i^N$	$W_{i+2N} = w_2$
$1 \le i \le 2N(N-1)$	$X_{i+2N+2^N} = r_3 c_i^2$	$W_{i+2N+2^N} = w_3$
$1 \le i \le 2^N$	$X_{i+2N+2^N+2N(N-1)} = r_4 c_i^N$	$W_{i+2N+2^N+2N(N-1)} = w_4$
$1 \le i \le N_1$	$X_{i+2N+2^N+2N(N-1)+2^N} = r_5 c_i^3$	$W_{i+2N+2^N+2N(N-1)+2^N} = w_5$
$1 \le i \le N2^N$	$X_{i+2N+2^N+2N(N-1)+2^N+N_1} = r_6 s_i^N$	$W_{i+2N+2^N+2N(N-1)+2^N+N_1} = w_6$
Central weight	$X_0 = \mathbf{0}$	$W_0 = w_0$

$$n = 2N + 2^N + 2N(N-1) + N_1 + 2^N + N2^N + 1\ ,\ \{N_1 = 4N(N-1)(N-2)/3\}$$



N. Adhurthi et al., "The Conjugate Unscented Transform - An Approach to Evaluate Multi-Dimensional Expectation Integrals,"

The moments Constraint equations up to 4^{nd} order are:

$$E[x_i^0] = 1 \Rightarrow \sum_{i=1}^{n} w_i = 1 \qquad E[x_i^2] = \frac{1}{3} \Rightarrow \sum_{i=1}^{n} w_i x_i^2 = \frac{1}{3}$$

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- Choosing 1 set of points on σ_i with weight w_i and 1 set of points on c_i^N with weight w_2.
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- $2Nw_1 + 2^N w_2 = 1 \qquad 2r_1^2 w_1 + 2^N r_2^2 w_2 = \frac{1}{3},$
- $2r_1^4 w_1 + 2^N r_2^4 w_2 = \frac{1}{5} \qquad 2^N r_2^4 w_2 = \frac{1}{9}$

The equations can be analytically solved as

$$r_1 = \sqrt{\frac{4+5N}{30}} \quad w_1 = \frac{40}{(4+5N)^2} \quad r_2 = \sqrt{\frac{4+5N}{-12+15N}} \quad w_2 = \frac{(4-5N)^2}{2^N(4+5N)^2}$$

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Conjugate Unscented Transformation

CUT4- Moments upto 4^{nd} order- Uniform pdf

The moments Constraint equations up to 4^{nd} order are:

$$E[x_i^0] = 1 \Rightarrow \sum_{i=1}^{n} w_i = 1 \qquad E[x_i^2] = \frac{1}{3} \Rightarrow \sum_{i=1}^{n} w_i x_i^2 = \frac{1}{3}$$

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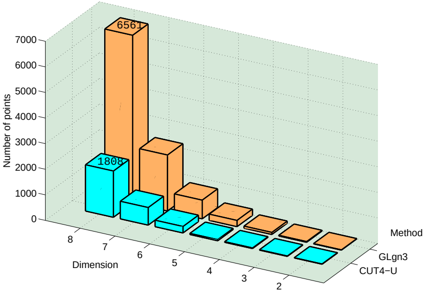
TABLE: CUT4-U rules

N	Point Selection	
2,3,4,5	$r_1\sigma_i$	$r_2 c_i^N$
	w_1	w_2
6	$r_1\sigma_i$	$r_2 c_i^4$
	w_1	w_2
7,8	$r_1\sigma_i$	$r_2 c_i^5$
	w_1	w_2



N. Adurthi, P. Singla and T. Singh, "Conjugate Unscented Transform Rules for Uniform Probability Density Functions," *2013 American Control Conference, Washington D.C., June17–19, 2013.*
N. Adurthi, P. Singla and T. Singh, "Conjugate Unscented Transformation: Applications to Estimation and Control," *ASME Journal of Dynamics, Measurements and Control, 2017.*

TABLE: CUT6-U rules

N	Point Selection			
2	$r_1\sigma_i$	$r_2 c_i^N$	$r_3 c_i^N$	-
	w_1	w_2	w_3	-
3,4	$r_1\sigma_i$	$r_2 c_i^N$	$r_3 c_i^2$	$r_4 c_i^N$
	w_1	w_2	w_3	w_4
5,6,7	$r_1\sigma_i$	$r_2 c_i^N$	$r_3 c_i^3$	$r_4 c_i^N$
	w_1	w_2	w_3	w_4
8,9	$r_1\sigma_i$	$r_2 c_i^N$	$r_3 c_i^4$	$r_4 c_i^N$
	w_1	w_2	w_3	w_4

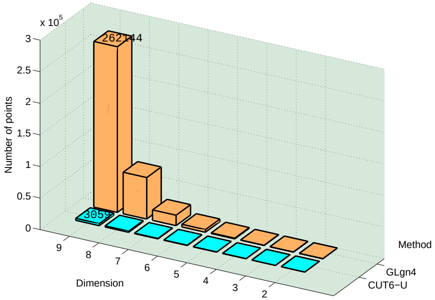


TABLE: CUT4-U rules

N	Point Selection					
2	$r_1\sigma_i$	$r_2c_i^2$	$r_3s_i^N(h)$	$r_4c_i^2$	-	-
	w_1	w_2	w_3	w_4	-	-
3	$r_1\sigma_i$	$r_2c_i^N$	$r_3c_i^2$	$r_4c_i^N$	$r_5s_i^N(h)$	-
	w_1	w_2	w_3	w_4	w_5	-
4	$r_1\sigma_i$	$r_2c_i^N$	$r_3c_i^2$	$r_4c_i^3$	$r_5s_i^N(h)$	$r_6c_i^N$
	w_1	w_2	w_3	w_4	w_5	w_6
5	$r_1\sigma_i$	$r_2c_i^N$	$r_3c_i^2$	$r_4c_i^3$	$r_5s_i^N(h)$	$r_6c_i^N$
	w_1	w_2	w_3	w_4	w_5	w_6

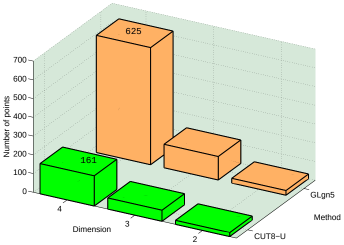


Table: Number of Points for 5^{th} order accurate cubature methods.

	GH		GHS		KPS		CUT4-G		HCKF-5	
Dim	$\sum\|w_i\|$	N	$\sum\|w_i\|$	N	$\sum\|w_i\|$	N	$\sum\|w_i\|$	N	$\sum\|w_i\|$	N
2	1	**<u>9</u>**	5	13	1	**<u>9</u>**	1	**<u>9</u>**	1	**<u>9</u>**
3	1	27	13	25	1	19	1	**<u>14</u>**	1	19
4	1	81	25	41	1	33	1	**<u>24</u>**	1	33
5	1	243	41	61	2.11	51	1	**<u>42</u>**	1.2041	51
6	1	729	61	85	3.67	**73**	1	<u>76</u>	1.37	**73**
7	1	2187	85	113	5.67	**99**	1	<u>142</u>	1.52	**99**
8	1	6561	113	145	8.11	**129**	1	<u>272</u>	1.64	**129**
9	1	19683	145	181	11	**163**	1	<u>530</u>	1.74	**163**

N ⇒ minimal number of quadrature points, <u>N</u> ⇒ minimal number of quadrature points with positive weight.
Gauss-Hermite Quadrature (GH), Gauss Hermite Smolyak quadrature (GHS), Kronrod Patterson Smolyak quadrature (KPS), High order cubature Kalman filter (HCKF)

TABLE: Number of Points for 7^{th} order accurate cubature points.

	GH		GHS		KPS		CUT6-G		HCKF-7	
Dim	$\sum \lvert w_i \rvert$	N	$\sum \lvert w_i \rvert$	N	$\sum \lvert w_i \rvert$	N	$\sum \lvert w_i \rvert$	N	$\sum \lvert w_i \rvert$	N
2	1	16	7	29	1	17	1	**13**	1	24
3	1	64	25	69	1.654	39	1	**27**	1	76
4	1	256	63	137	2.293	81	1	**49**	1	176
5	1	1024	129	241	2.931	151	1	**83**	1	340
6	1	4096	231	389	4.754	257	1	**137**	1.56	584
7	1	16384	377	589	7.614	**407**	1	423	2.1	924
8	1	65536	575	849	11.808	**609**	1	721	2.57	1376
9	1	262144	833	1177	17.63	**871**	1	1203	3.01	1956

N ⇒ minimal number of quadrature points, N ⇒ minimal number of quadrature points with positive weight.
Gauss-Hermite Quadrature (GH), Gauss Hermite Smolyak quadrature (GHS), Kronrod Patterson Smolyak quadrature (KPS), High order cubature Kalman filter (HCKF)

TABLE: Number of Points for 9^{th} order accurate cubature points.

	GH		GHS		KPS		CUT8-G	
Dim	$\sum\lvert w_i\rvert$	N	$\sum\lvert w_i\rvert$	N	$\sum\lvert w_i\rvert$	N	$\sum\lvert w_i\rvert$	N
2	1	25	9	53	1.485	37	1	**21**
3	1	125	41	165	2.6360	93	1	**59**
4	1	625	129	385	3.825	201	1	**161**
5	1	3125	321	781	4.95	401	1	**355**
6	1	15625	681	1433	6.07	749	1	**745**

N $\Rightarrow$ minimal number of quadrature points, N $\Rightarrow$ minimal number of quadrature points with positive weight.
Gauss-Hermite Quadrature (GH), Gauss Hermite Smolyak quadrature (GHS), Kronrod Patterson Smolyak quadrature (KPS), High order cubature Kalman filter (HCKF)

$$Q = \int 0.1 \sum_{i=1}^{6} x_i^8 \mathcal{N}(x:0,I)\,dx$$

TABLE: True Value of Integral=63.

Method	Q_{approx}	min(w)	$\sum \lvert w_i \rvert$	N
UT	205.8000	0.0714	1	13
$HCKF-5$	-28.8000	-0.0156	1.3750	73
$CUT4-G$	21.6000	0.0039	1	76
$CUT6-G$	60.5981	8.2885e-004	1	137
$CUT8-G$	63.0000	7.8492e-005	1	745
$GH3$	16.2000	2.1433e-005	1	729
$GH4$	48.6000	9.3219e-009	1	4096
$GH5$	63.0000	2.0353e-012	1	15625
$GH6$	63.0000	2.7871e-016	1	46656
$GH7$	63.0000	2.7162e-020	1	117649
$GH8$	63.0000	2.0397e-024	1	262144
$GHS3$	16.2000	-2.5000	61	85
$GHS4$	48.6000	-30.0000	231	389
$GHS5$	63.0000	-13.3333	681	1433
$GHS6$	63.0000	-90.3333	1683	4541
$GHS7$	63.0000	-34.9444	3653	12841
$GHS8$	63.0000	-199.3439	7183	33193

$$Q = \int cos(\|x\|_2)\mathcal{N}(x:0,I)\,dx$$

TABLE: True Value of Integral=-0.543583844.

Method	Q_{approx}	%err	min(w)	$\sum \|w_i\|$	N
UT	-0.6111	12.4130	0.0714	1	13
$HCKF-5$	-0.4635]	14.7285	-0.0156	1.3750	73
$CUT4-G$	-0.5492	1.0370	0.0039	1	76
$CUT6-G$	-0.5419	0.3013	8.2885e-004	1	137
$CUT8-G$	-0.5430	0.0995	7.8492e-005	1	745
$GH3$	-0.5162	5.0418	2.1433e-005	1	729
$GH4$	-0.5457	0.3918	9.3219e-009	1	4096
$GH5$	-0.5435	0.0229	2.0353e-012	1	15625
$GH6$	-0.5436	0.0011	2.7871e-016	1	46656
$GH7$	-0.5436	4.2494e-005	2.7162e-020	1	117649
$GH8$	-0.5436	1.4920e-006	2.0397e-024	1	262144
$GHS3$	-0.1910	64.8579	-2.5000	61	85
$GHS4$	-0.6023	10.7967	-30.0000	231	389
$GHS5$	-0.5370	1.2203	-13.3333	681	1433
$GHS6$	-0.5441	0.1026	-90.3333	1683	4541
$GHS7$	-0.5435	0.0068	-34.9444	3653	12841
$GHS8$	-0.5436	3.6928e-004	-199.3439	7183	33193

$$E[f(\mathbf{x})] = \int \mathcal{N}(\mathbf{x} : \mathbf{0}, \mathbf{P_i})(\sqrt{1+\mathbf{x}^T\mathbf{x}})^4 d\mathbf{x} \quad (1)$$

$$P_1 = \begin{bmatrix} 114.2595 & 90.1397 & 8.9751 \\ 90.1397 & 92.2504 & 29.1237 \\ 8.9751 & 29.1237 & 84.0908 \end{bmatrix}, \quad P_2 = 100I_{(10\times 10)} \quad (2)$$

TABLE: GH vs CUT4: % rel. error and no. of points

method	n_1	ε_1 % error	n_2	ε_2 % error
UT	7	51.55	21	8.32
MC	1000	21.84	1000	4.41
MC	10000	3.59	10000	0.35
MC	100000	0.64	100000	0.0027
GH2	8	52.36	1024	16.39
GH3	27	$4.89e-014$	59049	$7.23e-012$
CUT4	14	0	1044	$6.72e-012$

$$E[f(\mathbf{x})] = \int \mathcal{N}(\mathbf{x}:\mathbf{0},\mathbf{P_i})(\sqrt{1+\mathbf{x}^T\mathbf{x}})^6 d\mathbf{x} \tag{3}$$

$$P_3 = 100 I_{(4\times 4)} \quad P_4 = 100 I_{(9\times 9)} \tag{4}$$

Table: GH vs CUT6: % rel. error and no. of points

method	n_3	ε_3 % error	n_4	ε_4 % error
UT	9	47.8	19	30.02
MC	1000	11.46	1000	7.65
MC	10000	2.09	10000	2.3
MC	100000	0.53	100000	0.058
GH3	81	12.45	19683	4.18
GH4	256	$2.31e-013$	262144	$1.37e-009$
CUT6	49	$6.49e-013$	1203	$6.26e-009$

$$E[f(\mathbf{x})] = \int \mathcal{N}(\mathbf{x} : \mathbf{0}, \mathbf{P_i})(\sqrt{1+\mathbf{x}^T\mathbf{x}})^8 d\mathbf{x} \tag{5}$$

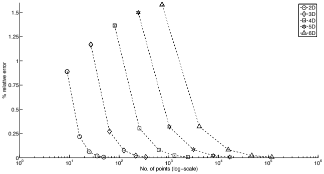
$$P_5 = 100I_{(5\times5)} \quad P_6 = 100I_{(6\times6)} \tag{6}$$

Table: GH vs CUT8: % rel. error and no. of points

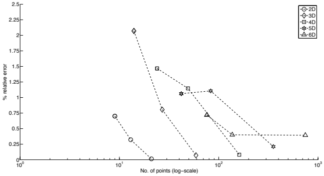
method	n_5	ε_5 % error	n_6	ε_6 % error
UT	11	68.74	13	64.19
MC	1000	19.514	1000	10.70
MC	10000	5.59	10000	5.17
MC	100000	0.43	100000	0.52
GH4	1024	3.45	4096	2.49
GH5	3125	$4.73e-012$	15625	$9.29e-012$
CUT8	355	$7.52e-012$	745	$6.63e-012$

NUMERICAL EXPERIMENTS

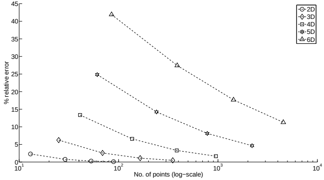
EXAMPLE: $E[(\sqrt{1+\mathbf{x}^T\mathbf{x}})^\alpha] = \int(\sqrt{1+\mathbf{x}^T\mathbf{x}})^\alpha \mathcal{N}(\mathbf{x}:\mathbf{0},0.1\mathbf{I})d\mathbf{x}$



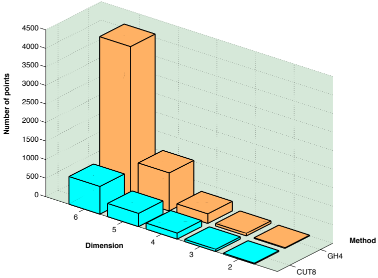
(a) Gauss-Hermite Convergence



(b) CUT Convergence

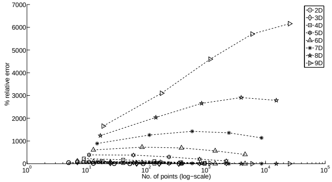


(c) Smolyak Scheme Convergence

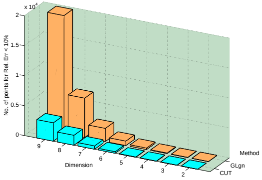


(d) Number of points required to achieve 0.5% Rel. error

$$E[(\sqrt{1+\mathbf{x}^T\mathbf{x}})^{-3}] = \frac{1}{2^N}\int_{-1}^{1}(\sqrt{1+\mathbf{x}^T\mathbf{x}})^{-3}d\mathbf{x}$$



(e) Smolyak Scheme Convergence



(f) Points required to achieve $\leq 10\%$ Relative error

- Longitudinal Dynamics Model:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\delta, \quad \delta = -\mathbf{K}\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} u_h \\ q_h \\ \theta_h \\ y \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} p_1 & p_2 & -g & 0 \\ p_3 & p_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} p_5 \\ p_6 \\ 0 \\ 0 \end{bmatrix}$$

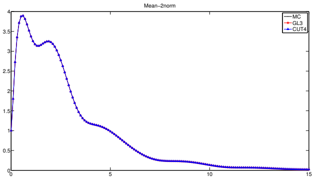
- Initial Conditions & Controller Gain:

$$\mathbf{x}_0 = [0.7929, -0.0466, -0.1871, 0.5780]^T$$
$$\mathbf{K} = [1.9890, -0.2560, -0.7589, 1.0]$$

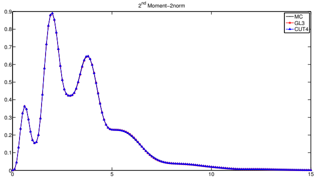
- Random Parameters: $\mathbf{p}_{lbd} = [-0.0488, 0.0013, 0.126, -3.3535]$ and $\mathbf{p}_{ubd} = [-0.0026, 0.0247, 2.394, -0.1765]$.

Table: 2-norm % Rel. Error

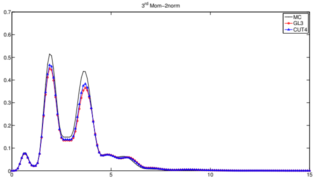
	GLgn3	**CUT4-U**	GLgn4	**CUT6-U**	GLgn5	**CUT8-U**
M_1	0.0176	**0.0171**	0.0161	**0.0160**	0.0160	**0.0160**
M_2	0.1150	**0.0216**	0.2338	**0.2741**	0.2576	**0.2599**
M_3	12.8793	**9.9926**	1.7471	**0.4916**	0.9488	**0.9122**
N	81	**24**	256	**65**	625	**161**



(g) 2-norm of Means

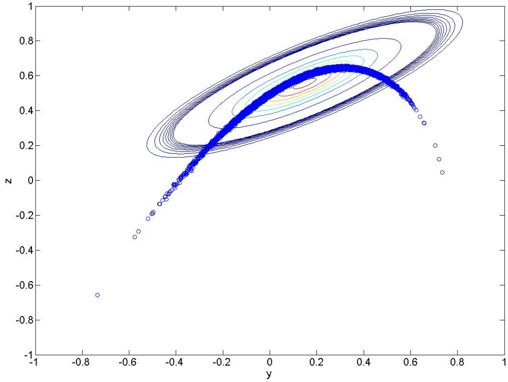


(h) 2-norm of 2^{nd} order Moments

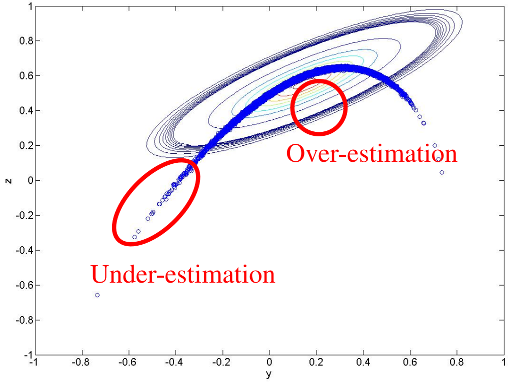


(i) 2-norm of 3^{rd} order Moments

Figure: *2-norm of the Moments vs time*



1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
z
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
y


Over-estimation
Under-estimation
z
y
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

- Moments only provide indirect information about the shape of the pdf.
- Need to construct the pdf, preferably expressed by an analytical expression.
- *Problem Statement*: Given a set of moments, find the best pdf that can represent these moments.

PRINCIPLE OF MAXIMUM ENTROPY

$$\max_{p(x)} -\int p(x) ln\Big(p(x)\Big)\,dx$$

s.t

$$\int x\,p(x)\,dx = M_1$$

$$\int x^2\,p(x)\,dx = M_2$$

$$\ldots$$

$$\int x^m\,p(x)\,dx = M_m$$

Using Lagrange Multipliers,

$$-\int p(x) ln\Big(p(x)\Big)\,dx + \sum_{i=1}^{m} \lambda_i \Big(\int x^i\,p(x)\,dx - M_i\Big)$$

By calculus of variation, the optimal pdf is given analytically as

$$p(x) = exp(\sum_{i=1}^{m} \lambda_i x^i)$$

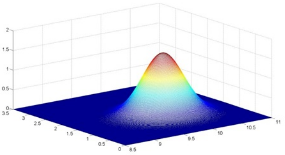
- Convex Optimization
- Can be solved by Gradient search methods.
- Has to be numerically preconditioned as there is $exp(polynomial)$.

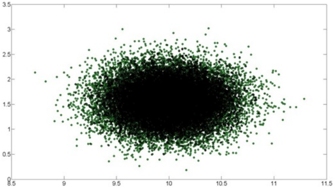
$$\int x\,exp(\sum_{i=1}^{m} \lambda_i x^i)\,dx = M_1$$

$$\int x^2\,exp(\sum_{i=1}^{m} \lambda_i x^i)\,dx = M_2$$

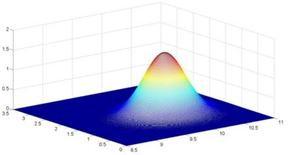
$$\ldots$$

$$\int x^m\,exp(\sum_{i=1}^{m} \lambda_i x^i)\,dx = M_m$$



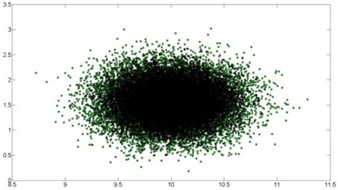


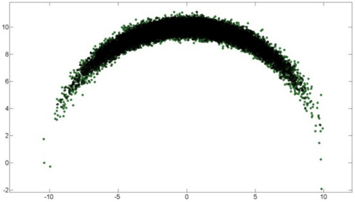
3.5
3
2.5
2
1.5
1
0.5
0
8.5
9
9.5
10
10.5
11
11.5

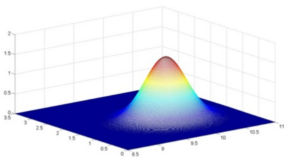


$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\boldsymbol{\theta} \\ r\sin\boldsymbol{\theta} \end{bmatrix}$$

$$E\begin{bmatrix} x \\ y \end{bmatrix} = \int \begin{bmatrix} r\cos\boldsymbol{\theta} \\ r\sin\boldsymbol{\theta} \end{bmatrix} N\left(\begin{bmatrix} r \\ \boldsymbol{\theta} \end{bmatrix} : \begin{bmatrix} \boldsymbol{\mu}_r \\ \boldsymbol{\mu}_\theta \end{bmatrix}, \boldsymbol{\Sigma}\right)$$

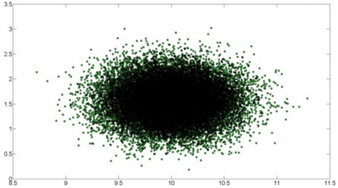


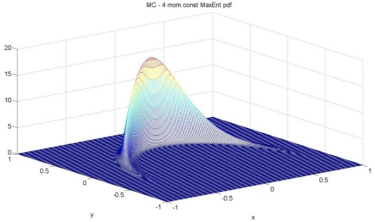




$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\boldsymbol{\theta} \\ r\sin\boldsymbol{\theta} \end{bmatrix}$$

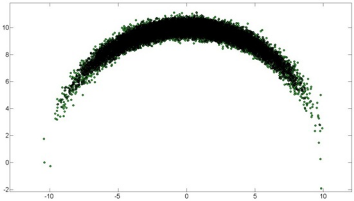
$$E\begin{bmatrix} x \\ y \end{bmatrix} = \int \begin{bmatrix} r\cos\boldsymbol{\theta} \\ r\sin\boldsymbol{\theta} \end{bmatrix} N\left(\begin{bmatrix} r \\ \boldsymbol{\theta} \end{bmatrix} : \begin{bmatrix} \boldsymbol{\mu}_r \\ \boldsymbol{\mu}_\theta \end{bmatrix}, \boldsymbol{\Sigma}\right)$$



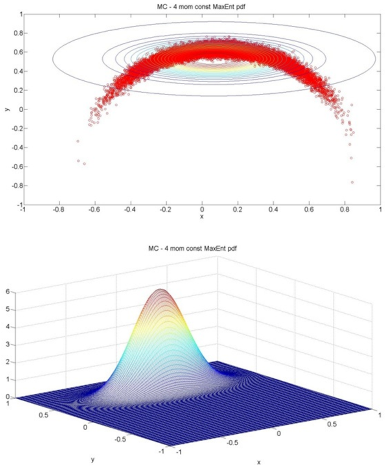


$$M_{(a,b)} = E[x^a y^b]$$

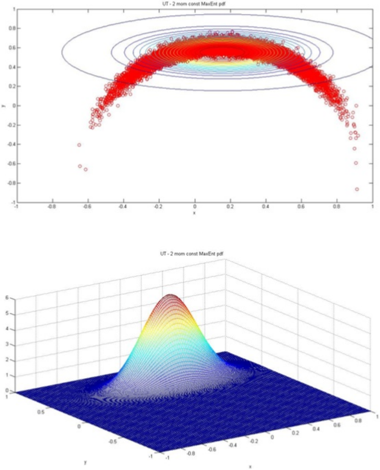
$$\forall\, a + b = d, \qquad d = \{1,2,\ldots\}$$



Using moments upto 2^{nd} order

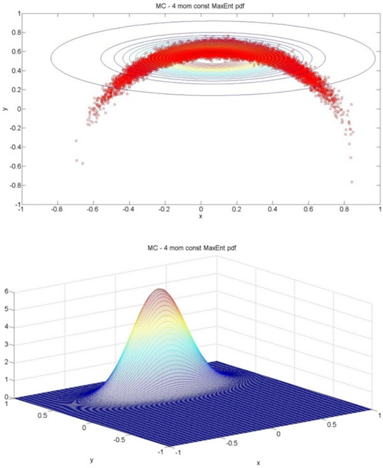


Monte Carlo: 30000 samples

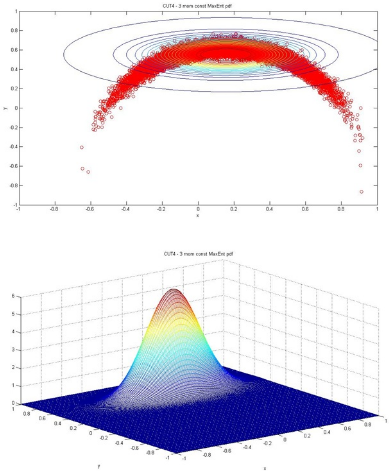


Unscented Transform:
5 points

Using moments upto 2^{nd} order

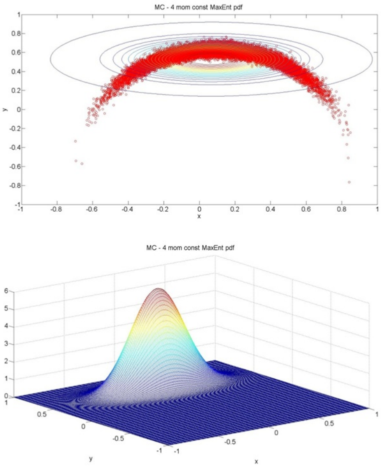


Monte Carlo: 30000 samples

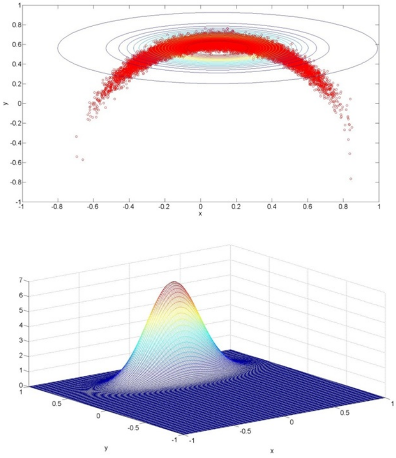


CUT4:
9 points

Using moments upto 2^{nd} order

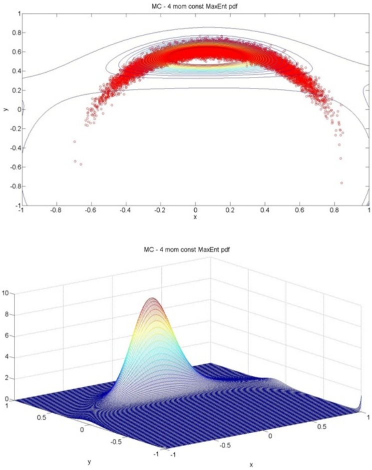


Monte Carlo: 30000 samples

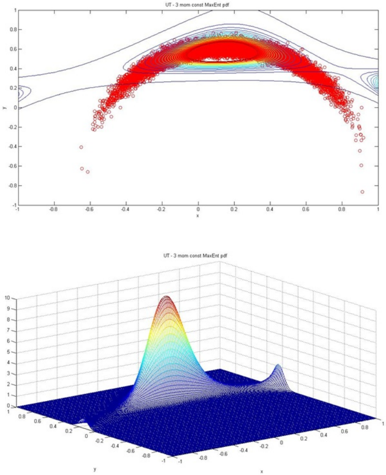


CUT8:
16 points

Using moments upto 3^{rd} order

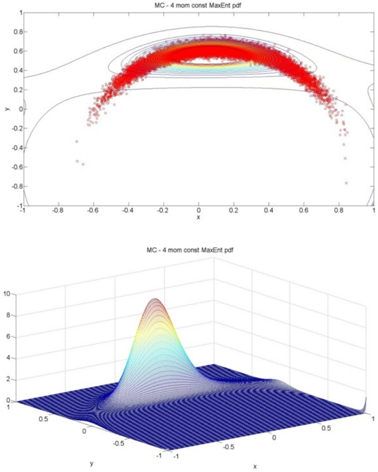


Monte Carlo: 30000 samples

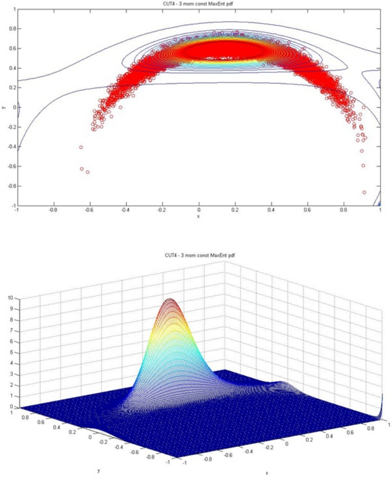


Unscented Transform:
5 points

Using moments upto 3^{rd} order

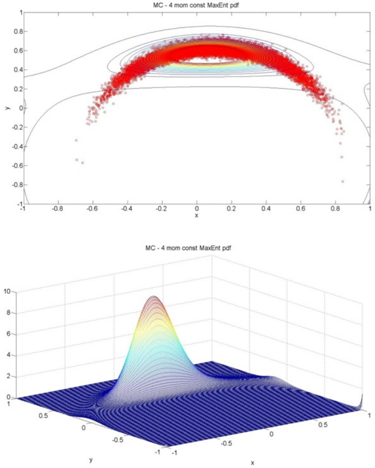


Monte Carlo: 30000 samples

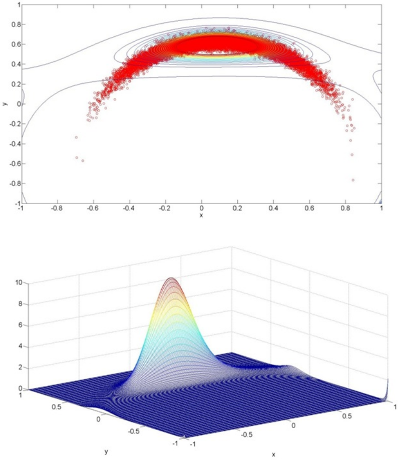


CUT4:
9 points

Using moments upto 3^{rd} order

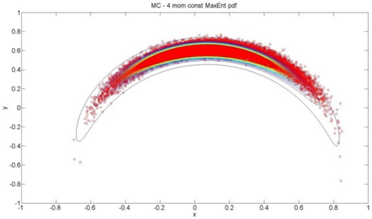


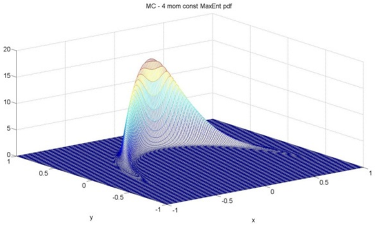
Monte Carlo: 30000 samples



CUT8:
16 points

Using moments upto 4^{th} order



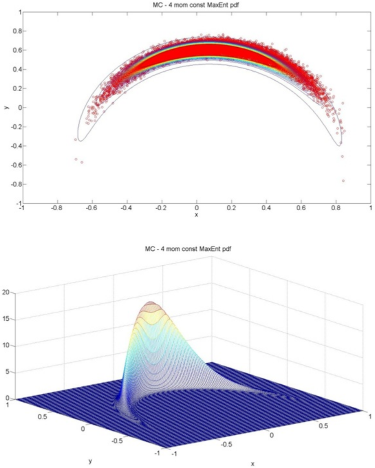


Monte Carlo: 30000 samples

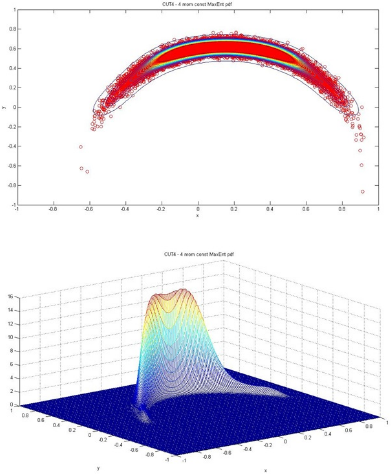
- Unscented Transform is only 3rd moment accurate
- Error in higher order moments
- Hence the pdf construction algorithm does not converge

Unscented Transform: 5 points

Using moments upto 4^{th} order

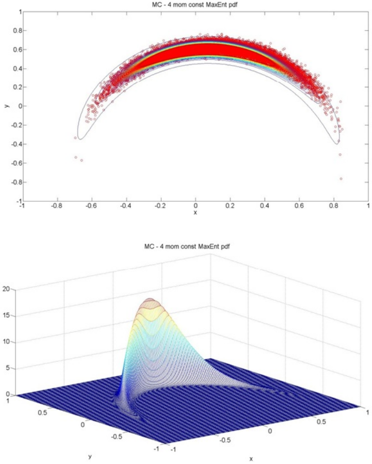


Monte Carlo: 30000 samples

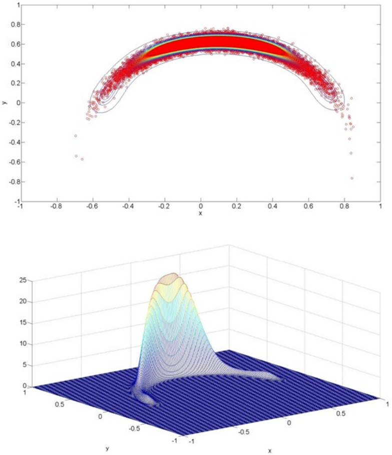


CUT4:
9 points

Using moments upto 4^{th} order



Monte Carlo: 30000 samples

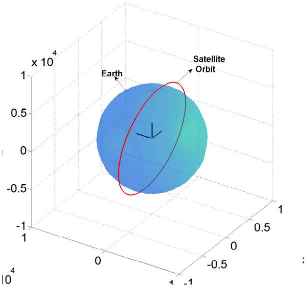


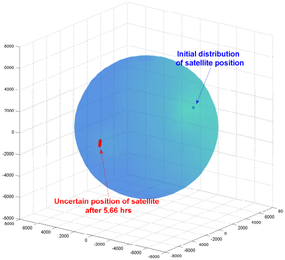
CUT8:
16 points

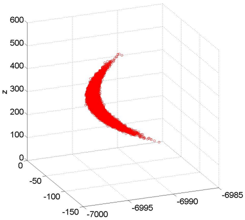
3D, there are

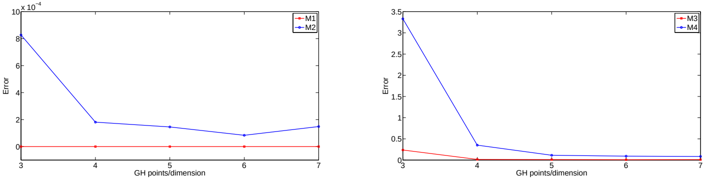
- 3 : 1^{st} order moments
- 6 : 2^{nd} order moments
- 10 : 3^{rd} order moments
- 15 : 4^{th} order moments
- A total of 34 lagrange multipliers to solve

- Reconstructing the pdf of Satellite position in full 3D.
- For visualization, this pdf is marginalized into 2D
- pdf contours compared against 200,000 MC runs









(a) All 1^{st} and 2^{nd} order moments

(b) All 3^{rd} and 4^{th} order moments

Figure: *Error compared to GH8 (8 points in each dimension)*

$$\ddot{x} = -\frac{\mu x}{r^3} + J_{2_x} + a_{D_x}, \quad J_{2_x} = -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(1-5\frac{z^2}{r^2}\right)\frac{x}{r}$$

$$\ddot{y} = -\frac{\mu y}{r^3} + J_{2_y} + a_{D_y}, \quad J_{2_y} = -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(1-5\frac{z^2}{r^2}\right)\frac{y}{r}$$

$$\ddot{z} = -\frac{\mu z}{r^3} + J_{2_z} + a_{D_z}, \quad J_{2_z} = -1.5J\left(\frac{\mu}{r^2}\right)\left(\frac{R_e}{r}\right)^2\left(3-5\frac{z^2}{r^2}\right)\frac{z}{r}$$

$$\mathbf{x}(0) \sim \mathcal{N}(\mu_0, \mathbf{P}_0) \quad \textit{where}$$

$$\mu_0 = [7000, 0, 0, 0, -1.0374090357, 7.4771288355]^T$$

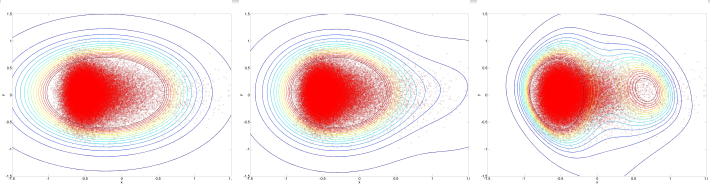
$$\mathbf{P}_0 = diag[0.01, 0.01, 0.01, 0.000001, 0.000001, 0.000001]$$

Table: RMSE in UT-CUT-GH5 (cartesian coordinates)

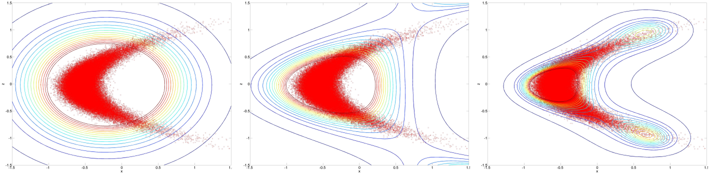
RMSE in	UT	CUT4	CUT6	CUT8	GH5	MC 100,000
e_1	1.18099856e-6	8.293251541e-10	4.2115520e-10	3.606078697e-10	2.809932915e-10	0.0330
e_2	0.186046892	0.0007910991	0.0011943952	0.0001853985	0.0001455770	0.8689
e_3	6.5478634011	0.2308841835	0.0107398551	0.009191179	0.012293992	4.6141
e_4	35.112930371	3.2570295399	0.205353106	0.0894419280	0.1143631243	4.6670

Two Body Problem

Reconstructing the position pdf with CUT4



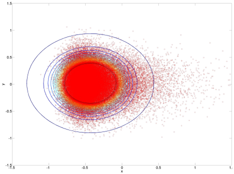
(a) (x, y): upto 2^{nd} order (b) (x, y): upto 3^{rd} order (c) (x, y): upto 4^{th} order



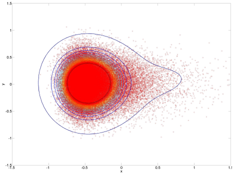
(d) (x, z): upto 2^{nd} order (e) (x, z): upto 3^{rd} order (f) (x, z): upto 4^{th} order

Figure: *CUT4: Marginalized pdf reconstruction using all 3D moments*[3]

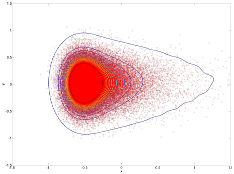
[3] N. Adurthi and P. Singla, "A Conjugate Unscented Transformation Based Approach for Accurate Conjunction Analysis," *AIAA Journal of Guidance, Control & Dynamics, 2015.*



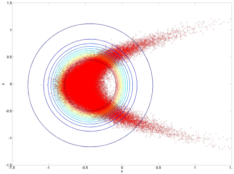
(a) (x,y): upto 2^{nd} order



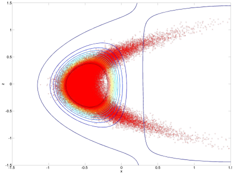
(b) (x,y): upto 3^{rd} order



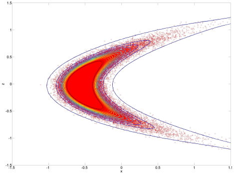
(c) (x,y): upto 4^{th} order



(d) (x,z): upto 2^{nd} order

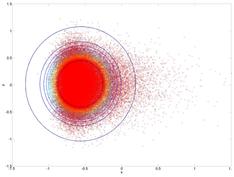


(e) (x,z): upto 3^{rd} order

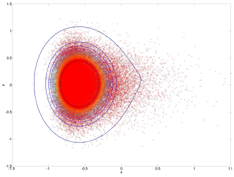


(f) (x,z): upto 4^{th} order

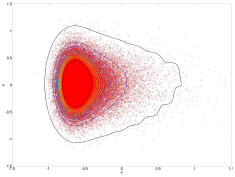
Figure: *CUT6: Marginalized pdf reconstruction using all 3D moments*



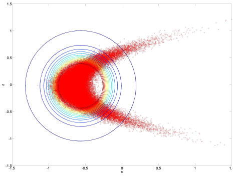
(a) (x,y): upto 2^{nd} order



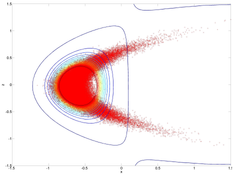
(b) (x,y): upto 3^{rd} order



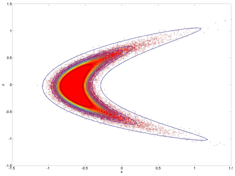
(c) (x,y): upto 4^{th} order



(d) (x,z): upto 2^{nd} order

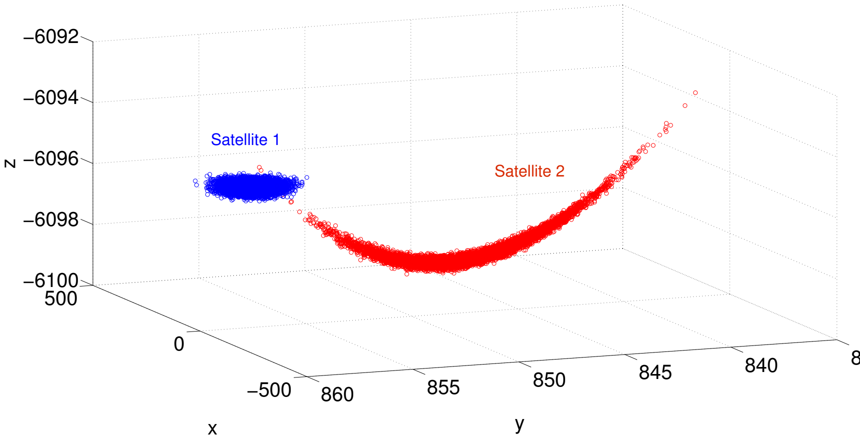


(e) (x,z): upto 3^{rd} order



(f) (x,z): upto 4^{th} order

Figure: *CUT8: Marginalized pdf reconstruction using all 3D moments*



Satellite 1
Satellite 2
z
−6092
−6094
−6096
−6098
−6100
500
0
−500
x
860
855
850
845
840
y

Miss distance $\boldsymbol{d} = \boldsymbol{x} - \boldsymbol{y}$

- $\boldsymbol{x}$: position of Satellite 1
- $\boldsymbol{y}$: position of Satellite 2

Probability that the objects are within a region Ω

sphere of radius R : $P\left(||\boldsymbol{d}|| < R\right)$

cube of half side length R : $P\left(||\boldsymbol{d}||_{max} < R\right)$

Monte Carlo simulations

$$\frac{No.\, of\ samples\ within\ R}{Total\ no.\, of\ samples}$$

Probability Density Approach

$$\int_{\Omega} p_d(\boldsymbol{x})d\boldsymbol{x}$$

If x and y are Gaussian random variables

$$x \sim N(x : \mu_x, P_x)$$
$$y \sim N(y : \mu_y, P_y)$$

then $d \sim N(d : \mu_d, P_d)$

$$\mu_d = \mu_x - \mu_y$$
$$P_d = P_x + P_y$$

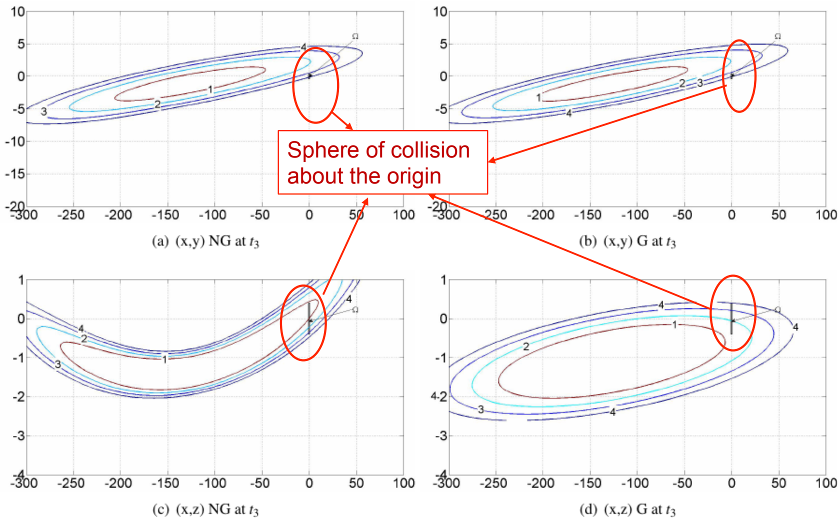
$$\boldsymbol{\rho} = \boldsymbol{x} - \boldsymbol{y}$$

Moments of ρ can be computed analytically, given the moments of x and y

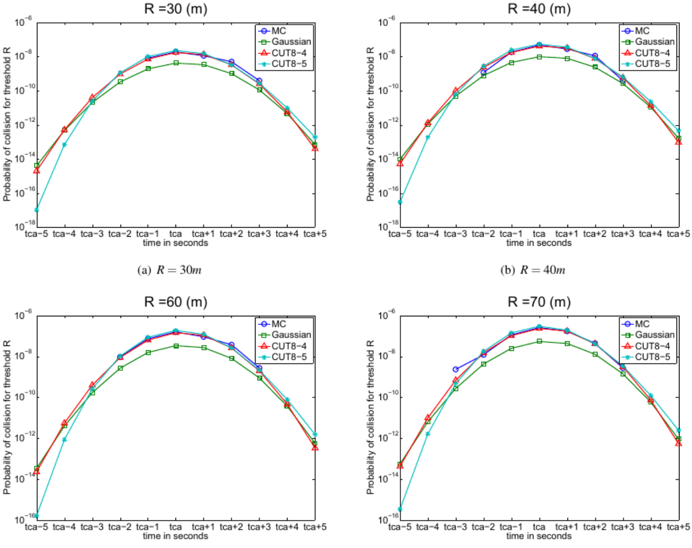
First Order	$E[\rho_1]$	$E[\rho_2]$	$E[\rho_3]$			
Second Order	$E[\rho_1^2]$	$E[\rho_2^2]$	$E[\rho_3^2]$	$E[\rho_1\rho_2]$	$E[\rho_2\rho_3]$	$E[\rho_1\rho_3]$
Third Order	$E[\rho_1^3]$	$E[\rho_2^3]$	$E[\rho_3^3]$	$E[\rho_1^2\rho_2]$	$E[\rho_2^2\rho_3]$	$E[\rho_1^2\rho_3]$
	$E[\rho_1\rho_2^2]$	$E[\rho_2\rho_3^2]$	$E[\rho_1\rho_3^2]$	$E[\rho_1\rho_2\rho_3]$		
Fourth Order	$E[\rho_1^4]$	$E[\rho_2^4]$	$E[\rho_3^4]$	$E[\rho_1^2\rho_2^2]$	$E[\rho_2^2\rho_3^2]$	$E[\rho_1^2\rho_3^2]$
	$E[\rho_1^2\rho_2\rho_3]$	$E[\rho_1\rho_2^2\rho_3]$	$E[\rho_1\rho_2\rho_3^2]$	$E[\rho_1^3\rho_2]$	$E[\rho_2^3\rho_3]$	$E[\rho_1^3\rho_3]$
	$E[\rho_1\rho_2^3]$	$E[\rho_2\rho_3^3]$	$E[\rho_1\rho_3^3]$			

Table 13. Case 2: RMSE in Computing Moments for Miss-Distance for Different Algorithms as Compared to GH7.

RMSE	GH4	GH5	CUT6	CUT8
e_1	4.5503e-011	1.858e-009	2.3286e-009	7.2916e-010
e_2	5.043e-008	3.8458e-007	8.2925e-006	7.8087e-008
e_3	1.5397e-005	6.5314e-005	0.0033037	1.327e-005
e_4	0.35904	0.010302	1.0176	0.0098918
N	4096	15625	137	745



Sphere of collision about the origin
(a) (x,y) NG at t_3
(b) (x,y) G at t_3
(c) (x,z) NG at t_3
(d) (x,z) G at t_3



(a) $R = 30m$ (b) $R = 40m$

(c) $R = 60m$ (d) $R = 70m$

Figure: *10 Million MC Runs vs. 1490 CUT Runs*

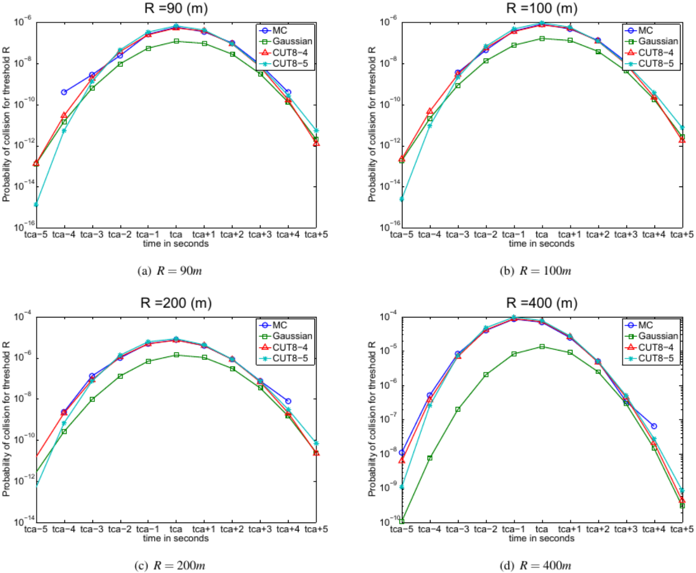


Figure: *10 Million MC Runs vs. 1490 CUT Runs*

NUMERICAL INTEGRATION

$$E[f(\mathbf{x})] = \int f(\mathbf{x})p(\mathbf{x})\,d\mathbf{x} \approx \sum_{i=1}^{N} w_i f(\mathbf{x}_i)$$

Monte Carlo Simulations

- Random points $\mathbf{x}_i \sim p(\mathbf{x})$
- Slow convergence : of order $\frac{1}{\sqrt{N}}$
- Large number of simulation are required

Gaussian Quadratures

- Fixed weights and points (roots of special polynomials)
- Faster convergence
- Tensor product of 1D points
- Exponential growth of points with dimension

Unscented Transform

- For n-Dimensions: just $2n+1$ points required for 3^{rd} degree polynomials
- $2n+1 \ll 2^n$ (Gaussian Quadratures)
- Error when used for higher degree polynomials

Smolyak Sparse Grid Quadrature

- Avoids tensor product by special sparse product of points
- Can contain negative weights

Non-product cubature rules- CUT

- Construct the points directly in the dimension required
- Avoid taking product of lower dimensional rules like Tensor Product and Sparse product
- Often results in a small fraction of point compared to Gaussian Quadratures and Smolyak quadrature