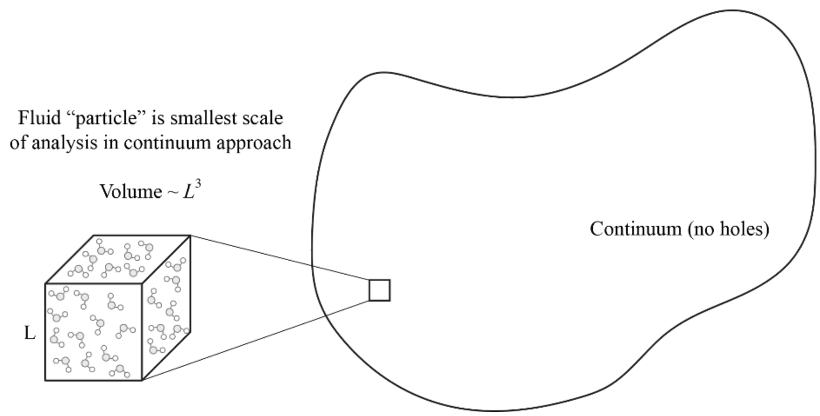
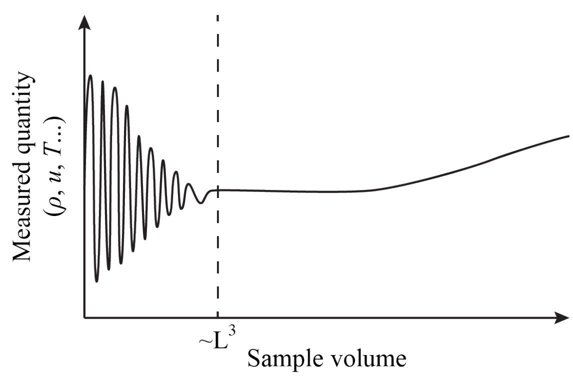
1. Fluid properties and concept of continuum

- Fluid: a substance that deforms continuously under an applied shear stress (e.g., air, water, upper mantle...)
- Not practical/possible to treat fluid mechanics at the molecular level!
- Instead, need to define a representative elementary volume (REV) to average quantities like velocity, density, temperature, etc. within a continuum
- Continuum: smoothly varying and continuously distributed body of matter – no holes or discontinuities



<u>1.1 What sets the scale of analysis?</u>

- Too small: bad averaging
- Too big: smooth over relevant scales of variability…



An obvious length scale L for ideal gases is the mean free path (average distance traveled by before hitting another molecule):

$$L = \frac{k_b}{4\pi\sqrt{2r^2}}\frac{T}{P} \tag{1}$$

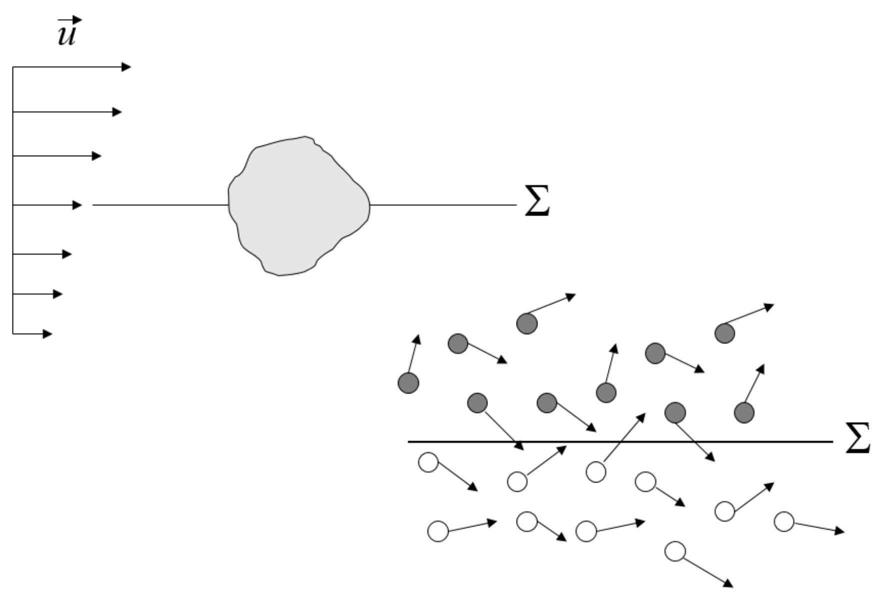
where k_b is the Boltzman constant, πr^2 is the effective cross sectional area of a molecule, T is temperature, and P is pressure.

Mean free path of atmosphere	
Sea level	$L \sim 0.1$ μm
z = 50 km	$L \sim 0.1$ mm
z = 150 km	$L \sim 1$ m

For liquids, not as straightforward to estimate L, but typically much smaller than for gas.

1.2 Consequences of continuum approach

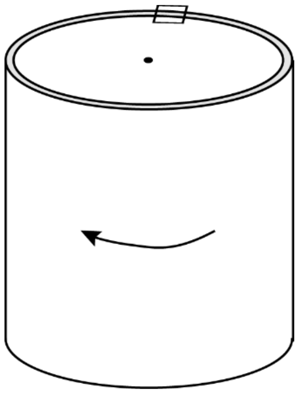
Consider a fluid particle in a flow with a gradient in the velocity field $\vec{u}$:

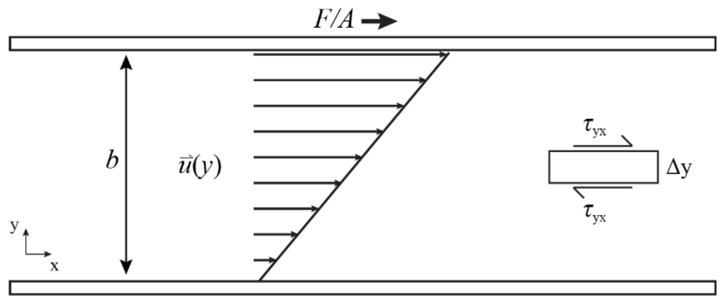


For real fluids, some "slow" molecules get caught in faster flow, and some "fast" molecules get caught in slower flow. This cannot be reconciled in continuum approach, so must be modeled. This is the origin of fluid shear stress and molecular viscosity. For gases, we can estimate viscosity from first principles using ideal gas law, calculating rate of momentum exchange directly. For liquids, experiments are needed...

1.3 Couette flow and fluid rheology

Maurice Couette in late 1800s performed a series of experiments using a cylinder viscometer to characterize fluid rheology and viscosity, and demonstrate validity of the "no slip" condition. Consider a small section of the flow that can be treated as flow between two plates, where a force F is applied to the upper plate (shear force per unit area = shear stress):





The velocity profile that develops depends on fluid rheology (i.e., constitutive equations that relate shear stress and strain rate). Here $\vec{u}$ indicates the velocity vector field (in Cartesian coordinates dependent on *x*, *y*, *z*, *t*), with the directional components u_x, u_y, and u_z corresponding to flow in the *x*, *y*, and *z* directions.

Experimental result (Newtonian fluid):

$$F/A \propto u_{max}/b \tag{2}$$

Newtonian fluid: linear relationship between applied shear stress and strain rate (in 1-D example above this is equal to the x-velocity gradient in y). For a given position in flow:

$$\boxed{\tau_{yx} = \mu \frac{\partial u_x}{\partial y}} \tag{3}$$

where τ_{yx} is the fluid shear stress acting in the x direction on the y-plane, and μ is the dynamic viscosity (units of $ML^{-1}T^{-1}$), a measure of fluid resistance to shear at the molecular level. We can also define the kinematic viscosity for convenience as:

$$\nu = \mu/\rho \tag{4}$$

Some common fluid viscosities (kg m^{-1} s^{-1})
Air: 2 x 10^{-5}
Water: 1 x 10^{-3}
Honey: 1 x 10^{1}
Upper mantle: 1 x 10^{20}

Non-Newtonian fluids: nonlinear relationship between shear stress and strain rate

- Bingham fluid: linear, but with critical shear stress τ_0 before deformation begins (debris flows)

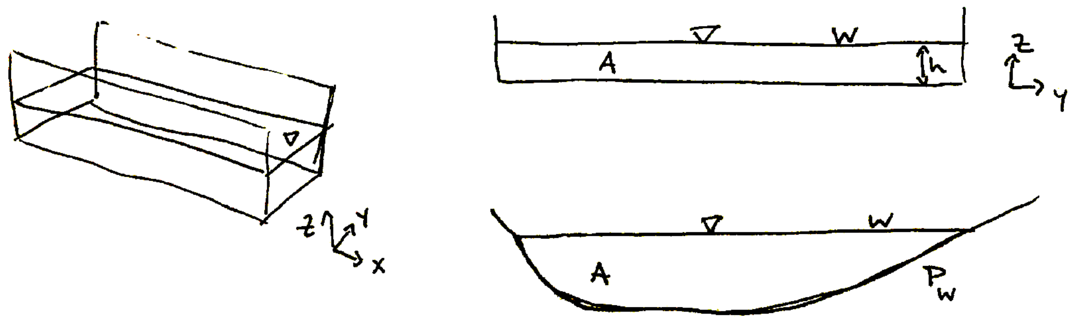
$$\tau_{yx} - \tau_0 = \mu_b \frac{\partial u_x}{\partial y} \tag{5}$$

- Glen's law: power-law relationship between shear stress and strain rate (ice flow in glaciers)

$$\frac{\partial u_x}{\partial y} = A\tau^n \quad (6)$$

where *A* depends on ice temperature, composition, etc, and *n* is typically ~3.

2. Basics of open-channel flow mechanics



Things we want to know:

- Vertical velocity profile (e.g., suspended sediment transport)
- Cross-sectional averaged velocity (e.g., discharge calculation)
- Bed shear stress (e.g., erosion and bedload transport)

For natural channels, typically use hydraulic radius instead of depth.

$$\boxed{R_h = A/P_w} \quad (7)$$

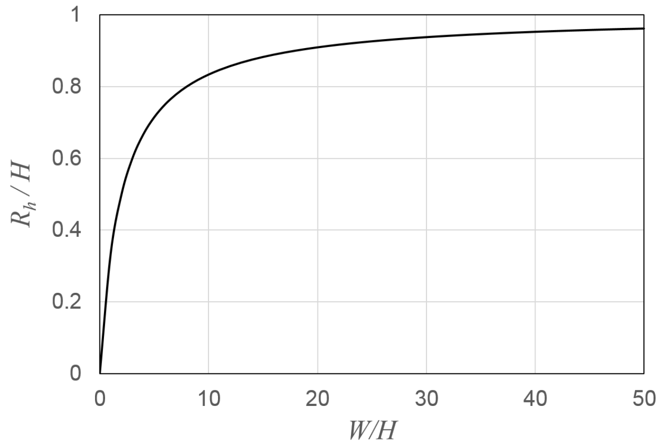
where *A* is cross-sectional area and P_w is the wetted perimeter. For rectangular channels, this becomes:

$$R_h = \frac{WH}{W+2H} \quad (8)$$

We can see that for "wide" channel (W>>H), $R_h = H$. What is wide? Non-dimensionalize to:

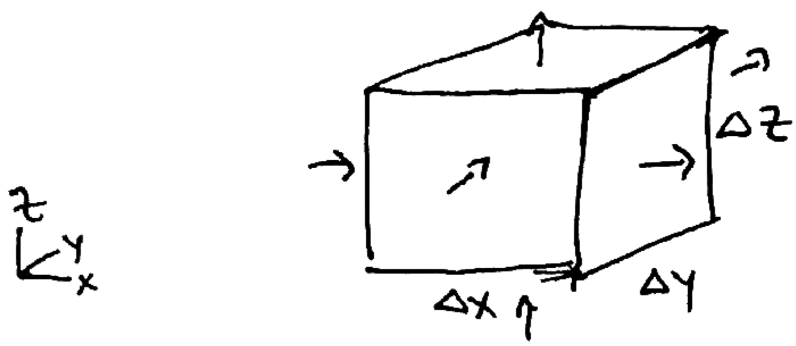
$$R_h/H = \frac{W/H}{W/H+2} \quad (9)$$

Plotting R_h/H vs W/H we can see that $R_h \sim H$ for $W/H > 20$.



2.1 Conservation of mass

For an arbitrary control volume V with density ρ:



Rate of change of mass within control volume = mass flux in – mass flux out:

LHS: rate of change of mass m within control volume:

$$\frac{\partial m}{\partial t} = \frac{\partial(\rho V)}{\partial t} = V\frac{\partial \rho}{\partial t} = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t} \quad (10)$$

RHS: Net mass flux into control volume:

$$\begin{aligned} &\rho u_x(x)\Delta y \Delta z - \rho u_x(x+\Delta x)\Delta y \Delta z + \\ &\rho u_y(y)\Delta x \Delta z - \rho u_y(y+\Delta y)\Delta x \Delta z + \\ &\rho u_z(z)\Delta x \Delta y - \rho u_z(z+\Delta z)\Delta x \Delta y \end{aligned} \quad (11)$$

Equating the above two expressions, and dividing through by the control volume $V = \Delta x \Delta y \Delta z$ results in:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\rho u_x(x+\Delta x) - \rho u_x(x)}{\Delta x} + \frac{\rho u_y(y+\Delta y) - \rho u_y(y)}{\Delta y} + \frac{\rho u_z(z+\Delta z) - \rho u_z(z)}{\Delta z}\right] \quad (12)$$

In the limit of small Δx, Δy, and Δz, the above equation simplifies to:

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z}\right] \tag{13}$$

and can be rearranged to form:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0 \tag{14}$$

which is the expression for conservation of mass in an Eulerian reference frame. We will also find it useful to track fluid properties in a Lagrangian reference frame, where an individual fluid particle is followed as it moves through space.

The material derivative of a property is indicated by the differential operator *D*/*Dt*, and defined (in this case for fluid density ρ):

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho \tag{15}$$

where the right hand terms refer to the change in field properties with time and the advection of a fluid particle through a spatially variable field (note that we will come back to this when deriving conservation of momentum to describe the concept of convective acceleration, which is the material derivative of velocity). Combining the above two equations (and applying the chain rule: $\nabla \cdot \rho \vec{u} = \rho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \rho$) results in the following expression for conservation of mass:

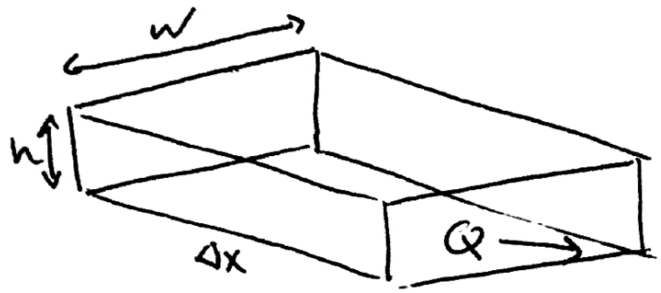
$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0 \tag{16}$$

If we assume that flow is incompressible (usually very reasonable), then the material derivative of density is zero, leading to the following form:

$$\boxed{\nabla \cdot \vec{u} = 0} \tag{17}$$

2.2 Conservation of mass (depth-averaged)

Often more useful to pick a larger control volume and look at depth-averaged quantities. Consider unidirectional flow a rectangular channel with uniform width. We will furthermore make the assumption that fluid density is uniform, which equates conservation of mass with conservation of volume.



For a rectangular cross section, the volumetric water flux Q is defined by:

$$\boxed{Q = \langle u_x \rangle W H} \tag{18}$$

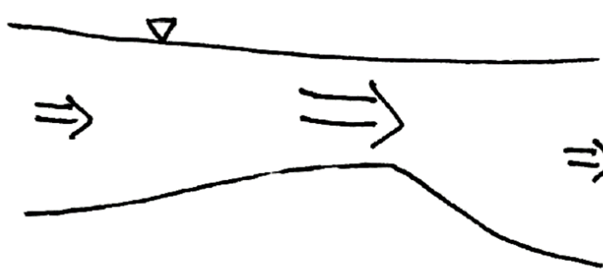
where $\langle u_x \rangle$ is the cross-sectional averaged downstream velocity. The statement of mass balance for the control volume can then be described by:

$$\frac{\partial (W H \Delta x)}{\partial t} = \langle u_x \rangle W H|_x - \langle u_x \rangle W H|_{x+\Delta x} \tag{19}$$

which after dividing both sides by $W\Delta x$ and taking the limit of small Δx becomes:

$$\boxed{\frac{\partial H}{\partial t} + \frac{\partial \langle u_x \rangle H}{\partial x} = 0} \tag{20}$$

Note also that this expression could equally be derived by 1D depth-averaging of the more general form $\nabla \cdot \vec{u} = 0$.



2.3 Conservation of momentum

Recall Newton's second law relating the sum of forces $\vec{F}$ acting on a body to the product of its mass m and acceleration $\vec{a}$:

$$\sum \vec{F} = m\vec{a} \tag{21}$$

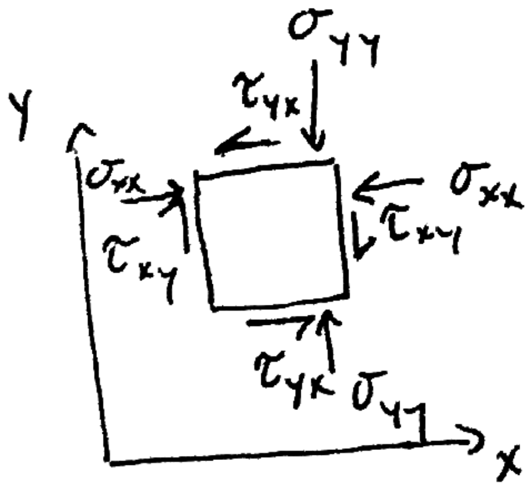
noting that the product $m\vec{a}$ is equal to the rate of change in momentum. When applied to fluids, the above equation becomes:

$$\sum \vec{F}_V = \rho \frac{D\vec{u}}{Dt} \tag{22}$$

where $\vec{F}_V$ indicates a force per unit volume acting on fluid parcel and $D\vec{u}/Dt$ is the material derivative of velocity, or the convective acceleration. This equation is an expression of the force balance acting on a fluid particle known as the Cauchy momentum equation. Recall that the material derivative of velocity can be expanded to:

$$\boxed{\frac{D\vec{u}}{Dt} = \frac{d\vec{u}}{dt} + \vec{u} \cdot \nabla \vec{u}} \tag{23}$$

The forces acting on the particle can be can be classified as either body forces or surface forces.



Body forces: In our case, the only external field acting on the particle is gravity (for ferromagnetic flows, magnetic field also needs to be taken into account). Recall that the gravitational force $\vec{F}_g = m\vec{g}$, where g is the gravitational acceleration vector. Thus, the gravitational force per volume becomes:

$$\vec{F}_{gv} = \rho \vec{g} \tag{24}$$

Surface (traction) forces: Surface forces include pressure and shear stresses that act in the form of traction on the surface of a fluid particle. Specifically, the surface force per volume $\vec{F}_{sv}$ can be described by:

$$\vec{F}_{sv} = \nabla \cdot \boldsymbol{\sigma}_{ij} \tag{25}$$

where $\boldsymbol{\sigma}_{ij}$ is the 3x3 stress tensor:

$$\boldsymbol{\sigma}_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{26}$$

This stress tensor can be split into a mean surface stress (pressure) field p, and a deviatoric stress tensor $\boldsymbol{\tau}_{ij}$ that characterizes shear stresses acting on the fluid:

$$\boldsymbol{\sigma}_{ij} = -p + \boldsymbol{\tau}_{ij} \tag{27}$$

$$p = -\frac{(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})}{3} \tag{28}$$

$$\boldsymbol{\tau}_{ij} = \begin{bmatrix} \sigma_{xx}+p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy}+p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz}+p \end{bmatrix} \tag{29}$$

This results in the following expression for the sum of fluid surface forces per volume:

$$\vec{F}_{sv} = -\nabla \mathrm{p} + \nabla \cdot \boldsymbol{\tau}_{ij} \tag{30}$$

Combining these equations, we get the following general expression for conservation of momentum:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla \mathrm{p} + \nabla \cdot \boldsymbol{\tau}_{ij} + \rho \vec{g} \tag{31}$$

This expression is not particularly useful because we are not typically able to quantify shear stresses directly (resulting in more unknowns than equations). Rather, we need a constitutive equation that relates the shear stress to fluid deformation. As we did earlier in 1D, we will assume a Newtonian fluid, which takes the general form:

$$\boxed{\boldsymbol{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \tag{32}$$

where the term in brackets is the symmetric component of the strain rate tensor (i.e., the component of strain that leads to deformation, as opposed to rotation). As an example, consider the *x* component of the viscous term:

$$\nabla \cdot \boldsymbol{\tau}_{ix} = \mu \nabla \cdot \left(\frac{\partial u_i}{\partial x} + \frac{\partial u_x}{\partial x_i}\right) \tag{33}$$

$$= 2\mu \frac{\partial^2 u_x}{\partial x^2} + \mu \frac{\partial^2 u_x}{\partial y^2} + \mu \frac{\partial^2 u_y}{\partial y \partial x} + \mu \frac{\partial^2 u_x}{\partial z^2} + \mu \frac{\partial^2 u_z}{\partial z \partial x} \tag{34}$$

$$= \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y \partial x} + \frac{\partial^2 u_z}{\partial z \partial x}\right) \tag{35}$$

$$= \mu \nabla^2 u_x + \mu \frac{\partial}{\partial x}(\nabla \cdot \vec{u}) = \mu \nabla^2 u_x \tag{36}$$

similar terms emerge for the *y* and *z* components (recall $\nabla \cdot \vec{u} = 0$ for incompressible flow), resulting in:

$$\nabla \cdot \boldsymbol{\tau}_{ij} = \mu \nabla^2 \vec{u} \qquad (37)$$

which we can plug into the momentum equation to get our final expression for conservation of momentum (Navier-Stokes equations) for an incompressible, Newtonian fluid:

$$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{g} \qquad (38)$$

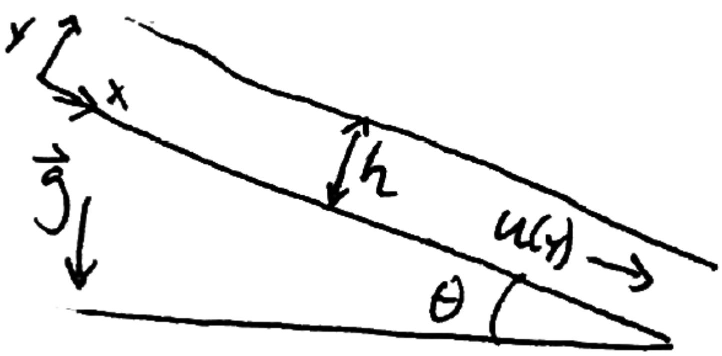
or alternatively:

$$\boxed{\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \vec{u} + \vec{g}} \qquad (39)$$

Note that this is a series of 3 second-order, non-linear partial differential equations (in *x*, *y*, and *z* directions for Cartesian coordinates), and when combined with conservation of mass (Eqn. 17) we have 4 equations with 4 unknowns (u_x, u_y, u_z, and *p*). The second (convective) term on the left hand side of Equation 39 is the non-linear term, and makes a general analytical solution fairly hopeless (as of 2016 one of six unsolved "Millenium Prize Problems"). We will explore a number of simplifications that enable analytical approximations for common flows, as well as use a 3D flow model (Delft3D) to solve more complicated problems numerically.

2.4 Application: flow down inclined plane

As a simple example, we will derive the velocity profile and bed shear stress for laminar flow down an inclined plane:



We will make a number of assumptions to help simplify things:

1. Incompressible flow: $\nabla \cdot \vec{u} = 0$
2. Newtonian fluid: $\boldsymbol{\tau}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$
3. Steady flow: $\frac{\partial \vec{u}}{\partial t} = 0$
4. Uniform flow: $u_y = u_z = 0$, *H* = constant, $\frac{\partial p}{\partial x} = 0$

5. Infinitely wide channel: $\frac{\partial u_x}{\partial z} = 0$

First, we apply conservation of mass using Equation 17:

$$\nabla \cdot \vec{u} = \frac{\partial u_x}{\partial \mathrm{x}} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0 \tag{40}$$

$$\frac{\partial u_x}{\partial \mathrm{x}} = 0 \tag{41}$$

Conservation of momentum in 1D becomes:

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial \mathrm{x}} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + g \sin\theta \tag{42}$$

Many of these terms cancel out with assumptions 1-5 to get:

$$\frac{d^2 u_x}{dy^2} = -\frac{\rho g \sin\theta}{\mu} \tag{43}$$

This is a second-order ordinary differential equation (u_x only varies in y). To solve we will need two boundary conditions. We will assume that there is no shear stress at the free surface, and that velocity goes to zero at the boundary (no slip condition):

$$\left.\frac{\mathrm{d}u_x}{dy}\right|_{y=H} = 0 \tag{44}$$

$$u_x|_{y=0} = 0 \tag{45}$$

Integrate with respect to y to get:

$$\frac{\mathrm{d}u_x}{dy} = -\frac{\rho g \sin\theta}{\mu} y + C_1 \tag{46}$$

Applying the first boundary condition we get:

$$\frac{\mathrm{d}u_x}{dy} = -\frac{\rho g \sin\theta}{\mu} y + \frac{\rho g \sin\theta}{\mu} h \tag{47}$$

Integrate once more to get:

$$u_x = -\frac{\rho g \sin\theta}{2\mu} y^2 + \frac{\rho g \sin\theta}{\mu} hy + C_2 \tag{48}$$

Applying the second boundary condition ($C_2 = 0$) we can simplify to:

$$\boxed{u_x = \frac{\rho g \sin\theta}{\mu}\left(hy - \frac{1}{2}y^2\right)} \tag{49}$$

which is the vertical velocity profile for laminar flow. We can also calculate the bed shear stress τ_b using the velocity gradient (Eqn. 47) evaluated at y = 0:

$$\tau_b = \mu \frac{\partial u_x}{\partial y}\bigg|_{y=0} = \mu\left(-\frac{\rho g \sin\theta}{\mu} y\bigg|_{y=0} + \frac{\rho g \sin\theta}{\mu} h\right) = \rho g h \sin\theta \qquad (50)$$

If the slope angle is low (less than about 10°), we can make the small angle approximation $S = \tan\theta \approx \sin\theta \approx \theta$ to get the familiar expression:

$$\tau_b = \rho g h S \qquad (51)$$

Although we used the velocity profile to determine the bed shear stress, it is very important to note that this relationship is independent of rheology (i.e., Eqns. 3, 5, 6). For any flow that is steady and uniform, the momentum equation simplifies to a force balance between the weight of the fluid and the resistance due to bed friction

Dimensionless Navier-Stokes equations

It is often useful to develop dimensionless forms of governing equations in order to better understand the controls on system behavior, and especially so when comparing model predictions with empirical data (either from the field or experiments). In order to do this, we need to define scaling factors for each of the dimensional variables present – in some cases this is a trivial process, and in others can be quite tricky. For the momentum equations (Eqn. 38 or 39), we can define the following scales:

$$\text{Length: } l_* = \frac{l}{L}, \nabla_* = L\nabla \qquad (52)$$

$$\text{Velocity: } \vec{u}_* = \frac{\vec{u}}{U} \qquad (53)$$

$$\text{Time: } t_* = \frac{t}{L/U} \qquad (54)$$

where the starred variables are dimensionless, and L and U are length and velocity scales. We also need to scale the pressure term, which is less straightforward, and can take two forms, depending on the dominant source of pressure fluctuations:

$$\text{Pressure (inertial): } p_* = \frac{p}{\rho U^2} \qquad (55)$$

$$\text{Pressure (viscous): } p_* = \frac{pL}{\mu U} \qquad (56)$$

We thus have two options for non-dimensionalizing the momentum equations, depending on the pressure scaling. If we use the inertial pressure scaling, Equation 38 becomes:

$$\frac{D\vec{u}_*}{Dt_*} = -\nabla_* p_* + \frac{\mu}{\rho U L}\nabla_*^2 \vec{u}_* + \frac{gL}{U^2}\hat{g} \tag{57}$$

where g is the magnitude of gravitational acceleration, and $\hat{g}$ is a unit vector in the direction of gravity. Two non-dimensional coefficients emerge, which can be defined as the Reynolds number (Re) and the Froude number (Fr):

$$Re = \frac{\rho U L}{\mu} \tag{58}$$

$$Fr = \frac{U}{\sqrt{gL}} \tag{59}$$

Combining Equations 57-59 results in:

$$\frac{D\vec{u}_*}{Dt_*} = -\nabla_* p_* + \frac{1}{Re}\nabla_*^2 \vec{u}_* + \frac{1}{Fr^2}\hat{g} \tag{60}$$

The Reynolds number can thus be thought of as the ratio of inertial forces to viscous forces (you can show this mathematically), and the Froude number is a measure of inertial forces relative to graviational forces. For cases when Re >> 1, Equation 60 can be simplified to:

$$\boxed{\frac{D\vec{u}_*}{Dt_*} = -\nabla_* p_* + \frac{1}{Fr^2}\hat{g}} \tag{61}$$

which is the non-dimensional form of Euler's equations for inviscid flow. The approximation of inviscid flow (i.e., no viscous forces) works well to describe pressure gradients away from the boundary layer where viscous forces dominate, and among other applications is useful for estimating lift forces on sediment in a river.

If instead we use the viscous pressure scaling (Eqn. 56), then the momentum equation becomes:

$$\frac{D\vec{u}_*}{Dt_*} = -\frac{1}{Re}\nabla_* p_* + \frac{1}{Re}\nabla_*^2 \vec{u}_* + \frac{1}{Fr^2}\hat{g} \tag{62}$$

For the case where Re << 1, we can simplify to:

$$\boxed{-\nabla_* p_* + \nabla_*^2 \vec{u}_* + \frac{Re}{Fr^2}\hat{g} = 0} \tag{63}$$

which is the non-dimensional form of the momentum equation for Stokes, or creeping flow approximation, which is useful for looking at particle settling velocities, for example. The Froude and Reynolds numbers emerge as the key parameters that describe the behavior of fluid momentum, and the implication from the above treatment is that flows with the same Froude and Reynolds numbers will behave the same – this is what is called dynamics similarity, and is the crux of simulating natural systems in downscaled laboratory flume experiments. Often it is impossible to achieve both Froude and Reynolds scaling and tradeoffs need to be evaluated (see Paola et al., 2009 for a good discussion of these challenges and compromises).

2.5 Turbulence

Observations of natural flows often reveals large-scale mixing due to turbulence (random, 3D velocity fluctuations due to fluid shear). Osborne Reynolds (late 1800s) performed a series of experiments using dye tracers to understand the transition from laminar to turbulent flow. This transition is governed by the dimensionless Reynolds number defined above (Eqn. 58), which describes the propensity for small disturbances to amplify by inertial forces or get dampened by viscous forces. For open channel flow, Re > 1000 typically is fully turbulent.

The net effect of turbulence is momentum diffusion in response to fluid shear, so how to incorporate this into the momentum equation (Eqn. 39)?

2.5.1 Reynolds averaging

Reynolds proposed decomposing velocity into its mean (deterministic) and fluctuating (stochastic) components, such that:

$$\vec{u} = \overline{\vec{u}} + \vec{u}' \qquad (64)$$

where $\overline{\vec{u}}$ is the time-averaged velocity, and $\vec{u}'$ is the deviation from this mean. Thus, we can substitute all of the velocities in the continuity and momentum equations (Eqn. 17 and 39) with the expression in Eqn. 64. Before we do this, need to define some math rules for the averaging. First, the mean of the sum is equal to the sum of the means, for example:

$$\overline{u_x + u_y} = \overline{u_x} + \overline{u_y} \qquad (65)$$

Second, the mean of a constant multiplied by velocity is simply the constant multiplied by the mean velocity:

$$\overline{a u_x} = a\overline{u_x} \qquad (66)$$

The mean of the product of two velocities, however, is not so trivial, and must be expanded:

$$\overline{u_x u_y} = \overline{\overline{u_x}\,\overline{u_y} + u_x'\overline{u_y} + u_y'\overline{u_x} + u_x' u_y'} \qquad (67)$$

Noting that the mean of the deviations is zero:

$$\overline{u_x'} = \overline{u_y'} = 0 \qquad (68)$$

the above expression can be simplified to:

$$\overline{u_x u_y} = \overline{u_x}\,\overline{u_y} + \overline{u_x' u_y'} \qquad (69)$$

where the term $\overline{u_x' u_y'}$ is the covariance of the two velocity components, and the source of much frustration.

Reynolds averaging of the continuity equation (Eqn. 17) results in an identical form, as there are no nonlinear terms.

$$\overline{\nabla \cdot \vec{u}} = \nabla \cdot \overline{\vec{u}} = 0 \tag{70}$$

However, when applying Reynolds averaging to the momentum equation, the convective term $\vec{u} \cdot \nabla \vec{u}$ requires careful treatment. For example, consider the convective term in the x-direction:

$$\vec{u} \cdot \nabla u_x = u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} + u_z \frac{\partial u_x}{\partial z} \tag{71}$$

Applying the chain rule, and assuming incompressibility ($\nabla \cdot \vec{u} = 0$), we can simplify to:

$$\vec{u} \cdot \nabla u_x = \frac{\partial u_x u_x}{\partial x} + \frac{\partial u_x u_y}{\partial y} + \frac{\partial u_x u_z}{\partial z} \tag{72}$$

Now, apply Reynolds averaging:

$$\overline{\frac{\partial u_x u_x}{\partial x} + \frac{\partial u_x u_y}{\partial y} + \frac{\partial u_x u_z}{\partial z}} = \frac{\partial \overline{u_x}\,\overline{u_x}}{\partial x} + \frac{\partial \overline{u_x}\,\overline{u_y}}{\partial y} + \frac{\partial \overline{u_x}\,\overline{u_z}}{\partial z} + \frac{\partial \overline{u_x' u_x'}}{\partial x} + \frac{\partial \overline{u_x' u_y'}}{\partial y} + \frac{\partial \overline{u_x' u_z'}}{\partial z} \tag{73}$$

$$= \overline{\vec{u}} \cdot \nabla \overline{u_x} + \frac{\partial \overline{u_x' u_x'}}{\partial x} + \frac{\partial \overline{u_x' u_y'}}{\partial y} + \frac{\partial \overline{u_x' u_z'}}{\partial z} \tag{74}$$

For three dimensions, we can generalize to:

$$\overline{\vec{u} \cdot \nabla \vec{u}} = \overline{\vec{u}} \cdot \nabla \overline{\vec{u}} + \nabla \cdot \overline{u_i' u_j'} \tag{75}$$

where the subscripts *i* and *j* indicate the components of a 3x3 tensor. Plugging back into the full momentum equation, and noting that the mean velocity does not change in time ($\partial \vec{u}/\partial t = 0$) we get:

$$\vec{u} \cdot \nabla \vec{u} + \nabla \cdot \overline{u_i' u_j'} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \nabla^2 \vec{u} + \vec{g} \tag{76}$$

where the u is the time-averaged velocity and the overbars on mean quantities have been dropped for clarity. We can furthermore rearrange to get:

$$\vec{u} \cdot \nabla \vec{u} = -\frac{1}{\rho} \nabla p + \frac{1}{\rho} \nabla \cdot \left(\mu \nabla \vec{u} - \rho \overline{u_i' u_j'} \right) + \vec{g} \tag{77}$$

Recall that $\mu \nabla \vec{u}$ is our model for the shear stress tensor (Eqn. 32) for an incompressible Newtonian fluid, which describes the efficiency of momentum tranfer via the molecular viscosity. The term $-\rho \overline{u_i' u_j'}$ is called the Reynolds stress tensor, and encapsulates the momentum transfer due to turbulent eddies. So… how to deal with this term? In order to "close" the Navier-Stokes equations, we need an additional expression or expressions to describe how the Reynolds stresses depend on the mean velocity field. This is the turbulence closure problem.

The simplest approach to this closure problem is to model the turbulent stresses in a similar fashion to how we modeled viscous stresses (Eqn. 32):

$$-\rho\overline{u_i'u_j'} = K\frac{\partial u_i}{\partial x_j} = K\nabla\vec{u} \tag{78}$$

where *K* is the eddy viscosity, and describes the efficiency momentum transport due to turbulent eddies similar to how μ describes the efficiency of momentum transport due to molecular fluctuations. Plugging this relationship into the momentum equation, we get:

$$\boxed{\vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\nabla\cdot(\mu\nabla\vec{u} + K\nabla\vec{u}) + \vec{g}} \tag{79}$$

It is crucial to note that in contrast to molecular viscosity, which is a property of the fluid, the eddy viscosity is a property of the flow. Additionally, for turbulent flows, $K \gg \mu$, except very close to the boundaries (in the laminar sublayer).

In the case of steady, uniform flow down an infinitely wide channel, Equation 79 simplifies to:

$$0 = \nabla\cdot(K\nabla\vec{u}) + \vec{g} \tag{80}$$

$$\frac{\partial}{\partial z}\left(K\frac{\partial u_x}{\partial z}\right) = -\rho g\sin\theta \tag{81}$$

Integrating both sides with respect to *z* gives:

$$K\frac{\partial u_x}{\partial z} = -\rho g z\sin\theta + C \tag{82}$$

Noting that the velocity gradient at a free surface must go to zero, the following boundary condition can be applied:

$$\left.\frac{\partial u_x}{\partial z}\right|_{z=H} = 0 \tag{83}$$

such that:

$$C = \rho g H\sin\theta = \tau_b \tag{84}$$

resulting in the following expression for bed shear stress in turbulent open channel flow:

$$\tau_b = K\frac{\partial u_x}{\partial z} \tag{85}$$

Prandtl mixing length hypothesis

Ludwig Prandtl in early 1900s proposed that the eddy viscosity K should scale with the size of the largest eddies in the flow, as well as the change in momentum over this length scale (i.e., the product of density, velocity gradient, and eddy size), giving in 1D:

$$K = \rho L^2 \frac{\partial u_x}{\partial z} \tag{86}$$

where L is the eddy length scale. Near the boundary ($z/H < 0.2$), Prantdl hypothesized that L is proportional to the distance from the wall:

$$L = \kappa z \tag{87}$$

where k is the von Karman constant ($\kappa = 0.4$ from experiments), resulting in:

$$\boxed{K = \rho \kappa^2 z^2 \frac{\partial u_x}{\partial z}} \tag{88}$$

Combining Equations 85 and 88 results in:

$$\tau_b = \rho \kappa^2 z^2 \left(\frac{\partial u_x}{\partial z}\right)^2 \tag{89}$$

which can be simplified to:

$$\frac{\partial u_x}{\partial z} = \frac{u_*}{\kappa} \frac{1}{z} \tag{90}$$

Integrating with respect to z results in:

$$u_x(z) = \frac{u_*}{\kappa} \ln z + C \tag{91}$$

Applying the boundary condition $u_x(z_0) = 0$, where $z_0 \ll H$ results in:

$$\boxed{u_x(z) = \frac{u_*}{\kappa} \ln \frac{z}{z_0}} \tag{92}$$

Equation 92 is known as "the law of the wall", but for natural rivers often describes well the full vertical velocity profile.