Incomplete Elliptic Integrals in Ramanujan’s Lost Notebook

Dan Schultz

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Introduction

On pages 51-53 of his lost notebook, Ramanujan recorded intriguing identities between $\eta$ functions and incomplete elliptic integrals. These identities take the form

$$\int_0^1 \text{product of } \eta \text{ functions } dq = \int U(q) L(q) \sqrt{1-k^2 \sin^2 \theta} \, d\theta$$

where $k \in \mathbb{C}$ is fixed.

We will see how these identities arise from complex function theory, especially the genesis of the elliptic curve on the r.h.s.

The proofs given here will not be reminiscent of Ramanujan's work on theta functions and modular equations. 

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We will see how these identities arise from complex function theory, especially the genesis of the elliptic curve on the r.h.s.

The proofs given here will not be reminiscent of Ramanujan’s work on theta functions and modular equations.
Notation

\( q = e^{2\pi i \tau}(x; q) \infty = (1 - x)(1 - xq) \ldots \)

\( \eta_a := \eta_a(\tau) = q^{a/24}(q^a; q^a) \infty \eta_a \),

\( n := \eta_a(\tau) = q^{(n-2)a/2}n(q^a; q^n) \infty (q^n - a; q^n) \infty (q^n; q^n) \infty \)

\( \text{SL}_2(\mathbb{Z}) = \{ (a \ b \\ c \ d) | ad - bc = 1 \} \)

\( \Gamma_0(N) = \{ M \in \text{SL}_2(\mathbb{Z}) \mid M \equiv \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) \mod N \} \)
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\[ \eta_{a,n} := \eta_{a,n}(\tau) = q^{(n-2a)^2/8n} (q^a; q^n)_\infty (q^{n-a}; q^n)_\infty (q^n; q^n)_\infty \]
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\[ SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| ad - bc = 1 \right\} \]
\[ \Gamma_0(N) = \left\{ M \in SL_2(\mathbb{Z}) \bigg| M \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\} \]
Examples

- level 15

\[ \int_{\eta}^{\infty} \frac{1}{\tau} \, d\tau = \frac{1}{5} \int_{\frac{\pi}{2}}^{\infty} \frac{\tan^{-1} \left( \frac{\eta}{\sqrt{5}} \right)}{\sqrt{1 - \epsilon - \frac{5}{3} \sin^2 \theta}} \, d\theta \]

where

\[ x = \eta_{3}^{2/15} \eta_{3}^{2/5}, \quad y = \eta_{5}^{1/5} \eta_{5}^{2/5}, \quad \epsilon = 1 + \sqrt{\frac{5}{2}} \]

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Examples

level 15

\[
\int_{i \infty}^{\tau} \eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{1}{5} \int \frac{2 \tan^{-1} \left( \frac{1}{\sqrt{5}} \right)}{2 \tan^{-1} \left( \frac{1}{\sqrt{5}} \sqrt{\frac{1-11x-x^2}{1+x-x^2}} \right)} \frac{d\theta}{\sqrt{1 - \frac{9}{25} \sin^2 \theta}}.
\]
Examples

- level 15

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\int_{i \infty}^{\tau} \eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{1}{5} \int 2 \tan^{-1} \left( \frac{1}{\sqrt{5}} \right) \frac{d\theta}{\sqrt{1 - \frac{9}{25} \sin^2 \theta}}.
\]

where

\[x = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3} \]

- level 10
Examples

▶ level 15

\[
\int_{i\infty}^{\tau} \eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{1}{5} \int 2 \tan^{-1} \left( \frac{1}{\sqrt{5}} \right) \frac{d\theta}{2 \tan^{-1} \left( \frac{1}{\sqrt{5}} \sqrt{1 - \frac{11x}{1 + x-x^2}} \right)} \sqrt{1 - \frac{9}{25} \sin^2 \theta}.
\]

where

\[ x = \frac{\eta_1 \eta_{15}}{\eta_3 \eta_5} \]

▶ level 10

\[
5^{3/4} \int_{i\infty}^{\tau} \eta_1^2 \eta_5^2 2\pi i d\tau = 2 \int_{\cos^{-1} \left( \sqrt{5y} \right)}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \left( e^{-5} \right)^{3/2} \sin^2 \theta}}
\]

\[ = \int_{0}^{\frac{\pi}{2}} 2 \tan^{-1} 5^{3/4} x \frac{d\theta}{\sqrt{1 - e^{-5} 5^{-3/2} \sin^2 \theta}} \]

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\int_{i\infty}^{\tau} \eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{1}{5} \int_{2}^{2\tan^{-1} \left( \frac{1}{\sqrt{5}} \right)} \left( \frac{1}{\sqrt{5}} \sqrt{\frac{1-11x-x^2}{1+x-x^2}} \right) \frac{d\theta}{\sqrt{1 - \frac{9}{25} \sin^2 \theta}}.
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\[
5^{3/4} \int_{i\infty}^{\tau} \eta_1^2 \eta_5^2 2\pi i d\tau = 2 \int_{\cos^{-1}}^{\pi/2} \sqrt{\epsilon^5 y} \frac{d\theta}{\sqrt{1 - \epsilon^{-5} 5^{-3/2} \sin^2 \theta}}
\]

where

\[ x = \frac{\eta_5^3}{\eta_1^3}, \quad y = \frac{\eta_{1,5}^5}{\eta_{2,5}^5}, \quad \epsilon = \frac{1 + \sqrt{5}}{2} \]
Examples

- level 14

\[ \int_{\tau}^{\infty} \eta_1 \eta_2 \eta_7 \eta_{14} \frac{\pi}{id \tau} = \int_{-\frac{7}{4}} \cos^{-1} c \cos^{-1} \left( 1 + x \right) d\theta \sqrt{1 - \frac{7}{2} \cdot 2^{-11} / 2} \sin^2 \theta. \]

\[ \int_{\tau}^{\infty} \eta_1 \eta_5 \eta_7 \eta_{35} \frac{\pi}{id \tau} = \int_{0}^{x} dx \sqrt{1 + x - x^2} \sqrt{1 - 5x - 9x^3 - 6x^5 - x^6}. \]

where \( x = \eta_1 \eta_35 \eta_5 \eta_7 \).
Examples

level 14

\[
\int_{i\infty}^{i\tau} \eta_1 \eta_2 \eta_7 \eta_{14} 2\pi i d\tau = 2^{-7/4} \int_{\cos^{-1} c}^{\cos^{-1} \frac{1+x}{1-x}} \frac{d\theta}{\sqrt{1 - 7 \cdot 2^{-11/2} c^{-2} \sin^2 \theta}}.
\]
Examples

- level 14

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\int_{i\infty}^{\tau} \eta_1 \eta_2 \eta_7 \eta_{14} 2\pi i d\tau = 2^{-7/4} \int_{\cos^{-1} c}^{\cos^{-1} c} \frac{d\theta}{\sqrt{1 - 7 \cdot 2^{-11/2} c^{-2}} \sin^2 \theta}.
\]

where

\[
x = \frac{\eta_1^4 \eta_{14}^4}{\eta_2^4 \eta_7^4}, \quad c = \frac{\sqrt{13 + 16\sqrt{2}}}{7}
\]

- level 35

...
Examples

- level 14

\[ \int_{i\infty}^{T} \eta_1 \eta_2 \eta_7 \eta_{14} 2\pi i d\tau = 2^{-7/4} \int_{\cos^{-1} c}^{\cos^{-1} c} \frac{d\theta}{\sqrt{1 - 7 \cdot 2^{-11/2} c^{-2} \sin^2 \theta}}. \]

where

\[ x = \frac{\eta_4^4 \eta_{14}^4}{\eta_2^4 \eta_7^4}, \quad c = \frac{\sqrt{13 + 16\sqrt{2}}}{7}. \]

- level 35

\[ \int_{i\infty}^{T} \eta_1 \eta_5 \eta_7 \eta_{35} 2\pi i d\tau = \int_{0}^{x} \frac{x \, dx}{\sqrt{1 + x - x^2} \sqrt{1 - 5x - 9x^3 - 6x^5 - x^6}}. \]
Examples

level 14

\[ \int_{\infty}^{\tau} \eta_1 \eta_2 \eta_7 \eta_{14} 2\pi i d\tau = 2^{-7/4} \int_{\cos^{-1} c}^{\cos^{-1} c \frac{1+x}{1-x}} \frac{d\theta}{\sqrt{1 - 7 \cdot 2^{-11/2} c^{-2} \sin^2 \theta}}. \]

where

\[ x = \frac{\eta_1^4 \eta_{14}^4}{\eta_2 \eta_7^4}, \quad c = \frac{\sqrt{13 + 16\sqrt{2}}}{7}. \]

level 35

\[ \int_{\infty}^{\tau} \eta_1 \eta_5 \eta_7 \eta_{35} 2\pi i d\tau = \int_{0}^{x} \frac{x \, dx}{\sqrt{1 + x - x^2} \sqrt{1 - 5x - 9x^3 - 6x^5 - x^6}} \]

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\[ x = \frac{\eta_1 \eta_{35}}{\eta_5 \eta_7}. \]
Examples’

These identities are established first by obtaining a differential equation, and then integrating both sides.
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▶ level 15
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level 15

$$\eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{dx}{\sqrt{(x^2 - x - 1)(x^2 + 11x - 1)}}$$
These identities are established first by obtaining a differential equation, and then integrating both sides.

**level 15**

\[ \eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{dx}{\sqrt{(x^2 - x - 1)(x^2 + 11x - 1)}} \]

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\[ x = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3} \]

**level 10**
Examples’

These identities are established first by obtaining a differential equation, and then integrating both sides.

- **level 15**

  \[ \eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{dx}{\sqrt{(x^2 - x - 1)(x^2 + 11x - 1)}} \]

  where

  \[ x = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3} \]

- **level 10**

  \[ \eta_1^2 \eta_5^2 2\pi i d\tau = \frac{dy}{\sqrt{y(1 - 11y + y^2)}} \]

  \[ = \frac{2dx}{\sqrt{1 + 22x^2 + 125x^4}} \]
Examples’

These identities are established first by obtaining a differential equation, and then integrating both sides.

▶ level 15

$$\eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{dx}{\sqrt{(x^2 - x - 1)(x^2 + 11x - 1)}}$$

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▶ level 10

$$\eta_1^2 \eta_5^2 2\pi i d\tau = \frac{dy}{\sqrt{y(1 - 11y + y^2)}}$$

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where

$$x = \frac{\eta_5^3}{\eta_1^3}, \quad y = \frac{\eta_{1,5}^5}{\eta_{2,5}^5}$$
Examples’

- level 14

\[
\eta_1 \eta_2 \eta_7 \eta_{14} \pi \text{id} \tau = dx \sqrt{1 - \frac{1}{2} x + \frac{1}{4} x^2 - \frac{1}{4} x^3 + x^4}
\]

where

\[
x = \eta_4 \eta_{14} \eta_2 \eta_7
\]

\[
\eta_1 \eta_5 \eta_7 \eta_{35} \pi \text{id} \tau = x \ dx \sqrt{1 + x - \frac{1}{2} x^2} \sqrt{1 - \frac{5}{2} x - \frac{9}{4} x^3 - \frac{6}{4} x^5 - x^6}
\]

where

\[
x = \eta_1 \eta_{35} \eta_5 \eta_7
\]
Examples’

- level 14

\[ \eta_1 \eta_2 \eta_7 \eta_{14} 2\pi i d\tau = \frac{dx}{\sqrt{1 - 14x + 19x^2 - 14x^3 + x^4}} \]
Examples’

- level 14

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\eta_1 \eta_2 \eta_7 \eta_{14} 2\pi i d\tau = \frac{dx}{\sqrt{1 - 14x + 19x^2 - 14x^3 + x^4}}
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- level 35
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level 35

\[ \eta_1 \eta_5 \eta_7 \eta_{35} 2 \pi i d\tau = \frac{x \ dx}{\sqrt{1 + x - x^2} \sqrt{1 - 5x - 9x^3 - 6x^5 - x^6}} \]
Examples’

- level 14

\[ \eta_1 \eta_2 \eta_7 \eta_{14} 2\pi i d\tau = \frac{dx}{\sqrt{1 - 14x + 19x^2 - 14x^3 + x^4}} \]

where

\[ x = \frac{\eta_1^4 \eta_{14}^4}{\eta_2^4 \eta_7^4} \]

- level 35

\[ \eta_1 \eta_5 \eta_7 \eta_{35} 2\pi i d\tau = \frac{x \, dx}{\sqrt{1 + x - x^2} \sqrt{1 - 5x - 9x^3 - 6x^5 - x^6}} \]

where

\[ x = \frac{\eta_1 \eta_{35}}{\eta_5 \eta_7} \]
Converting Back to Legendre’s Normal Form

Elliptic integrals with a quartic polynomial under the square root can be converted to Legendre form via the equation

\[ \int_0^X \sqrt{(x-r_1)(x-r_2)(x-r_3)(x-r_4)} \, dx = 2\sqrt{(r_3-r_1)(r_2-r_4)} \int \frac{1}{\sqrt{(r_2-r_4)(X-r_1)(r_3-r_1)(r_2-r_4)}} \, d\theta \sqrt{1-(r_3-r_4)(r_2-r_1)(r_3-r_1)(r_2-r_4) \sin^2 \theta}, \]

Elliptic integrals with a cubic polynomial under the square root can be converted via

\[ \int_0^X \sqrt{(x-r_1)(x-r_2)(x-r_3)} \, dx = 2\sqrt{r_3-r_1} \int \frac{1}{\sqrt{(X-r_1)(r_2-r_1)}} \, d\theta \sqrt{1-(r_2-r_1)(X-r_1)(r_3-r_1) \sin^2 \theta}. \]
Converting Back to Legendre’s Normal Form

Elliptic integrals with a quartic polynomial under the square root can be converted to Legendre form via the equation

\[
\int_0^X \frac{dx}{\sqrt{(x-r_1)(x-r_2)(x-r_3)(x-r_4)}}
\]

\[
= \frac{2}{\sqrt{(r_3-r_1)(r_2-r_4)}} \int_0^{\sin^{-1}\sqrt{\frac{(r_2-r_4)(X-r_1)}{(r_2-r_1)(X-r_4)}}} d\theta \sqrt{1 - \frac{(r_3-r_4)(r_2-r_1)}{(r_3-r_1)(r_2-r_4)} \sin^2 \theta},
\]

Elliptic integrals with a cubic polynomial under the square root can be converted via

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Elliptic integrals with a quartic polynomial under the square root can be converted to Legendre form via the equation

\[ \int_0^X \frac{dx}{\sqrt{(x - r_1)(x - r_2)(x - r_3)(x - r_4)}} = \frac{2}{\sqrt{(r_3 - r_1)(r_2 - r_4)}} \int_0^{\sin^{-1} \sqrt{(r_2 - r_4)(x - r_1)/(r_2 - r_1)(x - r_4)}} d\theta \sqrt{1 - \left(\frac{r_3 - r_4}{r_3 - r_1}\right)\left(\frac{r_2 - r_1}{r_2 - r_4}\right)\sin^2 \theta}, \]

Elliptic integrals with a cubic polynomial under the square root can be converted via

\[ \int_0^X \frac{dx}{\sqrt{(x - r_1)(x - r_2)(x - r_3)}} = \frac{2}{\sqrt{r_3 - r_1}} \int_0^{\sin^{-1} \sqrt{\frac{x - r_1}{r_2 - r_1}}} d\theta \sqrt{1 - \left(\frac{r_2 - r_1}{r_3 - r_1}\right)\sin^2 \theta}. \]
Transformations of Legendre’s Normal Form

degree two Landen transformation

\[ 2 + \int_0^\Theta d\theta \sqrt{1 - 4t(1 + t)^2}\sin^2 \theta = \int \tan^{-1}\left(\sin 2\Theta \cos 2\Theta + t\right) d\theta \sqrt{1 - t^2}\sin^2 \theta, \]

degree three transformation

\[ 3 + 2t \int_0^\Theta d\theta \sqrt{1 - t\left(2 + t(1 + 2t)^3\right)}\sin^2 \theta = \int 2 \tan^{-1}\left(tan \frac{\Theta}{2}\right)(t + 2) \cos \Theta - t + 1(t + 2) \cos \Theta + t - 1) d\theta \sqrt{1 - t^3(2 + t)^3}\sin^2 \theta, \]

and the double angle formula

\[ 2 \int_0^\Theta d\theta \sqrt{1 - t \sin^2 \theta} = \int 2 \tan^{-1}\left(tan \Theta \sqrt{1 - t \sin^2 \Theta}\right) d\theta \sqrt{1 - t \sin^2 \theta}. \]
Transformations of Legendre’s Normal Form

degree two Landen transformation

\[
\frac{2}{1 + t} \int_{0}^{\Theta} \frac{d\theta}{\sqrt{1 - \frac{4t}{(1+t)^2} \sin^2 \theta}} = \int_{0}^{\tan^{-1} \left( \frac{\sin 2\Theta}{\cos 2\Theta + t} \right)} \frac{d\theta}{\sqrt{1 - t^2 \sin^2 \theta}},
\]
Transformations of Legendre’s Normal Form

degree two Landen transformation

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\frac{2}{1 + t} \int_{0}^{\Theta} \frac{d\theta}{\sqrt{1 - \frac{4t}{(1+t)^2} \sin^2 \theta}} = \int_{0}^{\tan^{-1}\left(\frac{\sin 2\Theta}{\cos 2\Theta + t}\right)} \frac{d\theta}{\sqrt{1 - t^2 \sin^2 \theta}},
\]

the degree three transformation

\[
\frac{3}{1 + 2t} \int_{0}^{\Theta} \frac{d\theta}{\sqrt{1 - t \left(\frac{2+t}{1+2t}\right)^3 \sin^2 \theta}} = \int_{0}^{2 \tan^{-1}\left(\tan\left(\frac{\Theta}{2}\right) \frac{(t+2) \cos \Theta - t+1}{(t+2) \cos \Theta + t-1}\right)} \frac{d\theta}{\sqrt{1 - t^3 \left(\frac{2+t}{1+2t}\right) \sin^2 \theta}},
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Transformations of Legendre’s Normal Form

degree two Landen transformation

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the degree three transformation

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\]

and the double angle formula

\[
2 \int_{0}^{\Theta} \frac{d\theta}{\sqrt{1 - t \sin^2 \theta}} = \int_{0}^{2 \tan^{-1}\left(\tan \Theta \sqrt{1 - t \sin^2 \Theta}\right)} \frac{d\theta}{\sqrt{1 - t \sin^2 \Theta}}.
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On Ramanujan’s Elliptic Integrals and Modular Identities
S. Raghavan, and S. S. Rangachari
Proof Example
\[ \frac{1}{2\pi i} \frac{dx}{d\tau} = \eta_1 \eta_3 \eta_5 \eta_{15} \sqrt{(x^2 - x - 1)(x^2 + 11x - 1)} \]
Proof Example

\[ \frac{1}{2\pi i} \frac{dx}{d\tau} = \eta_1 \eta_3 \eta_5 \eta_{15} \sqrt{(x^2 - x - 1)(x^2 + 11x - 1)} \]

\[ x = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3} \]
Proof Example

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\frac{1}{2\pi i} \frac{dx}{d\tau} = \eta_1 \eta_3 \eta_5 \eta_{15} \sqrt{(x^2 - x - 1)(x^2 + 11x - 1)}
\]

\[
x = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3}
\]

\[
R = \frac{\eta_1^2 \eta_5^2}{\eta_3^2 \eta_{15}^2}, \quad P = \frac{\eta_1^6}{\eta_5^6}, \quad Q = \frac{\eta_3^6}{\eta_{15}^6}
\]
Proof Example

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\frac{1}{2\pi i} \frac{dx}{d\tau} = \eta_1 \eta_3 \eta_5 \eta_{15} \sqrt{(x^2 - x - 1)(x^2 + 11x - 1)}
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\[
x = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3}
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R = \frac{\eta_1^2 \eta_5^2}{\eta_3^2 \eta_{15}^2}, \quad P = \frac{\eta_1^6}{\eta_5^6}, \quad Q = \frac{\eta_3^6}{\eta_{15}^6}
\]

\[
R + 5 + \frac{9}{R} = \frac{1}{x} - x
\]
Proof Example

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\frac{1}{2\pi i} \frac{dx}{d\tau} = \eta_1 \eta_3 \eta_5 \eta_{15} \sqrt{(x^2 - x - 1)(x^2 + 11x - 1)}
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\[
x = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3}
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R = \frac{\eta_1^2 \eta_5^2}{\eta_3^2 \eta_{15}^2}, \quad P = \frac{\eta_1^6}{\eta_5^6}, \quad Q = \frac{\eta_3^6}{\eta_{15}^6}
\]

\[
R + 5 + \frac{9}{R} = \frac{1}{x} - x
\]

\[
P + \frac{125}{P} = R - 4 + \frac{135}{R} + \frac{486}{R^2} + \frac{729}{R^3}
\]
Proof Example

\[ \frac{1}{2\pi i} \frac{dx}{d\tau} = \eta_1 \eta_3 \eta_5 \eta_{15} \sqrt{(x^2 - x - 1)(x^2 + 11x - 1)} \]

\[ \chi = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3} \]

\[ R = \frac{\eta_1^2 \eta_5^2}{\eta_3^2 \eta_{15}^2}, \quad P = \frac{\eta_1^6}{\eta_5^6}, \quad Q = \frac{\eta_3^6}{\eta_{15}^6} \]

\[ R + 5 + \frac{9}{R} = \frac{1}{\chi} - \chi \]

\[ P + \frac{125}{P} = R - 4 + \frac{135}{R} + \frac{486}{R^2} + \frac{729}{R^3} \]

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Proof Example

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\[
Q + \frac{125}{Q} = R^3 + 6R^2 + 15R - 4 + \frac{9}{R}
\]

\[
1 + 6 \sum_k \sigma_1(k) q^k - 30 \sum_k \sigma_1(k) q^{5k} = \sqrt{\frac{\eta_1^{12} + 22\eta_1^6 \eta_5^6 + 125\eta_5^{12}}{\eta_1^2 \eta_5^2}}
\]
We will obtain these formulas by

1. Finding $N$ such that the modular curve of level $N$ is an elliptic curve $E/C$

2. Constructing the invariant differential for $E/C$ as cusp form $\times 2\pi id\tau$

3. Constructing the invariant differential for $E/C$ from a function $x$ of order 2 on $E/C$

4. Equating these two representation of the invariant differential gives the main differential equation

Ramanujan has correctly identified the cusp form and correctly identified the function $x$ of order 2.
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Ramanujan has

- correctly identified the cusp form
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The modular curve $\mathbb{H}/\Gamma_0(N)$

Theorem

For any $N \in \mathbb{N}$, $\mathbb{H}/\Gamma_0(N)$ can be made into a compact Riemann surface by the addition of certain cusps in $\mathbb{Q} \cup \{i\infty\}$.

The fundamental domain of $\mathbb{H}/\Gamma_0(N)$ consist of $[\Gamma_0(1) : \Gamma_0(N)]$ translates of the fundamental domain of $\mathbb{H}/\Gamma_0(1)$.

$q = e^{2\pi i \tau}$ may be used as a local variable at the cusp $i\infty$ of $\mathbb{H}/\Gamma_0(N)$. 

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The modular curve $\mathbb{H}/\Gamma_0(N)$

a given

\[
\begin{pmatrix}
 a & b \\
 c & d \\
\end{pmatrix} \in \Gamma_0(N)
\]

acts on $\tau \in \mathbb{H}$ by

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}
\]
The modular curve $\mathbb{H}/\Gamma_0(N)$

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The modular curve \( \mathbb{H}/\Gamma_0(N) \)

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The modular curve $\mathbb{H}/\Gamma_0(N)$

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may be used as a local variable at the cusp $i\infty$ of $\mathbb{H}/\Gamma_0(N)$. 
The genus of \( \mathbb{H}/\Gamma_0(N) \)
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$$2g - 2 = \frac{1}{6} N \prod_{p | N, p \text{ prime}} (1 + p^{-1})$$
The genus of $\mathbb{H}/\Gamma_0(N)$

\[ 2g - 2 = \frac{1}{6} N \prod_{p|N, p \text{ prime}} (1 + p^{-1}) \]

\[- \sum_{c|N} \phi(\gcd(c, N/c)) \]
The genus of $\mathbb{H}/\Gamma_0(N)$

$$2g - 2 = \frac{1}{6}N \prod_{p|N, p \text{ prime}} (1 + p^{-1})$$

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$$- \frac{1}{2} \begin{cases} 
\prod_{q|N, q \text{ prime}} \left(1 + \left(\frac{-1}{q}\right)\right), & 4 \nmid N \\
0, & 4|N 
\end{cases}$$
The genus of $\mathbb{H}/\Gamma_0(N)$

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$$- \frac{2}{3} \begin{cases} 0, & 9 \nmid N \\ \prod_{q|N, q \text{ prime}} \left(1 + \left(\frac{-3}{q}\right)\right), & 9 \mid N \end{cases}$$
The genus of $\mathbb{H}/\Gamma_0(N)$

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Key Fact: when $g = 1$, $\mathbb{H}/\Gamma_0(N)$ is isomorphic to some elliptic curve over $\mathbb{C}$. 
An elliptic curve over $\mathbb{C}$ is either of

- $\mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$ with $\text{im}(\omega_2/\omega_1) > 0$
- $y^2 = 4x^3 - g_2x - g_3$
An elliptic curve over $\mathbb{C}$ is either of

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The $\wp$ function provides the conversion between the two

\[
x = \wp(z) \\
y = \wp'(z)
\]
An elliptic curve over $\mathbb{C}$ is either of

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The $\wp$ function provides the conversion between the two

\[
x = \wp(z) \\
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\]

The integral on either side of Ramanujan’s identities provides the explicit isomorphism between $\mathbb{H}/\Gamma_0(N)$ and $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ for some $\omega_1$ and $\omega_2$.

\[
z = \int_{i\infty}^{\tau} f(\tau)2\pi id\tau
\]

where $f(\tau)$ is a cusp form.
Elliptic curve (cont)

Theorem

On an elliptic curve $E/\mathbb{C}$, the space of holomorphic differentials is one-dimensional ($= \mathbb{C} \Omega$).

Examples:

▶ $\mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}) \Omega = dz$

▶ $y^2 = 4x^3 - g_2x - g_3 \Omega = dx y = dx \sqrt{4x^3 - g_2x - g_3}$

▶ $H/\Gamma_0(\mathbb{N})$ with $g = 1 \Omega = \text{cuspform}$

$\times 2\pi i \delta \tau = \text{cuspform} \times dq q$

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Theorem

On an elliptic curve $E/\mathbb{C}$, the space of holomorphic differentials is one dimensional ($= \mathbb{C}\Omega$).
**Theorem**

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**Examples:**
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Theorem

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Examples:

- $\mathbb{C}/(\omega_1 \mathbb{Z} + \omega_2 \mathbb{Z})$
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Examples:

- $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$
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- $y^2 = 4x^3 - g_2x - g_3$
  $\Omega = \frac{dx}{y} = \frac{dx}{\sqrt[3]{4x^3 - g_2x - g_3}}$

- $\mathbb{H}/\Gamma_0(N)$ with $g = 1$
  $\Omega = \text{cuspform} \times 2\pi i d\tau = \text{cuspform} \times \frac{dq}{q}$
It is also possible to construct $\Omega$ from an arbitrary function $x$ of order 2 on the elliptic curve. Theorem

If $x$ is a function of order 2 on an elliptic curve with poles $p_1$ and $p_2$, then

$$\Omega = \frac{dx}{\sqrt{(x - x(r_1))(x - x(r_2))(x - x(r_3))(x - x(r_4))}}$$

is holomorphic. Here, the $r_i$ are the four solutions to $r_i \oplus r_i = p_1 \oplus p_2$ on the elliptic curve with a group law $\oplus$. If $x(r_i) = \infty$, the corresponding factor is omitted.
It is also possible to construct $\Omega$ from an arbitrary function $x$ of order 2 on the elliptic curve.

**Theorem**

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We seek to construct a function of order 2 on $\mathbb{H}/\Gamma_0(N)$.

Definition

A function $f$ on $\mathbb{H}/\Gamma_0(N)$ is called an $\eta$ quotient if $\text{ord}_\tau = \tau_0(f) = 0$ for $\tau_0 \in \mathbb{H}$.

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We seek to construct a function of order 2 on $\mathbb{H}/\Gamma_0(N)$. 
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**Definition**

A function $f$ on $\mathbb{H}/\Gamma_0(N)$ is called an $\eta$ quotient if

\[
\text{ord}_{\tau = \tau_0}(f) = 0 \quad \text{for} \quad \tau_0 \in \mathbb{H}
\]

\[
\text{ord}_{\tau = \frac{1}{c}}(f) \quad \text{depends only on the denominator} \quad c
\]
η Quotients on $\mathbb{H}/\Gamma_0(N)$

**Theorem**

All $\eta$ quotients on $\mathbb{H}/\Gamma_0(N)$ are of the form

$$
\prod_{l|N} \eta(l\tau)^{r_l}, \quad r_l \in \mathbb{Z}
$$

where

$$
\sum_{l|N} r_l = 0
$$

$$
\sum_{l|N} lr_l \in 24\mathbb{Z}
$$

$$
\sum_{l|N} \frac{N}{l} r_l \in 24\mathbb{Z}
$$

$$
\prod_{l|N} l^{r_l} \in \mathbb{Z}^2
$$
Orders of $\eta$ Quotients on $\mathbb{H}/\Gamma_0(N)$

Using the transformation law of the $\eta$ function

$$
\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon_{abcd} \sqrt{c\tau + d} \eta(\tau), \quad \epsilon_{abcd}^{24} = 1
$$

one can show that

$$
\text{ord}_{\tau = \frac{l}{c}}(\eta(l\tau)) = \begin{cases} 
gcd(l,c)^2 \cdot \frac{N/c}{24l} & \text{if } c \neq 0, \\
\frac{l}{24} & \text{if } c = 0.
\end{cases}
$$

Thus, using a list of inequivalent cusps of $\mathbb{H}/\Gamma_0(N)$ we can write down $\eta$ quotients and verify that they have order 2.
Using the transformation law of the $\eta$ function

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one can show that

$$\text{ord}_{\tau = \frac{c}{c}}(\eta(l\tau)) = \begin{cases} \frac{\gcd(l,c)^2 N/c}{24l} \frac{N/c}{\gcd(c, N/c)}, & c \neq 0 \\ \frac{l}{24}, & c = 0 \end{cases}.$$ 

$$\text{ord}_{\tau = \frac{c}{c}} \prod_{l \mid N} \eta(l\tau)^{r_l} = \sum_{l \mid N} r_l \text{ord}_{\tau = \frac{c}{c}} \eta(l\tau)$$
Orders of $\eta$ Quotients on $\mathbb{H}/\Gamma_0(N)$

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Thus, using a list of inequivalent cusps of $\mathbb{H}/\Gamma_0(N)$ we can write down $\eta$ quotients and verify that they have order 2.
Cusp forms on $\mathbb{H}/\Gamma_0(N)$

We can also write

$$\Omega = f(\tau)^2 \pi i d\tau$$

where, under $\Gamma_0(N)$,

$$f(a\tau + b\tau + c\tau + d) = (c\tau + d)^2 f(\tau) = dq q$$

has simple poles at all of the cusps.

We need $\text{ord}_\tau = \cdot c(f(\tau)) > 0$.

Such an $f$ can usually be constructed as an $\eta$ quotient

$$f(\tau) = \prod l | N \eta(l\tau)^{r_l}, \quad r_l \in \mathbb{Z}$$

where $\sum l | N r_l = 4$. 
We can also write

\[ \Omega = f(\tau)2\pi i d\tau \]
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\[
\begin{align*}
    f \left( \frac{a\tau + b}{c\tau + d} \right) &= (c\tau + d)^2 f(\tau) \\
    d \left( \frac{a\tau + b}{c\tau + d} \right) &= (c\tau + d)^{-2} d\tau
\end{align*}
\]
We can also write
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\[ 2\pi id\tau = \frac{dq}{q} \] has simple poles at all of the cusps.
Cusp forms on $\mathbb{H}/\Gamma_0(N)$

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Cusp forms on $\mathbb{H}/\Gamma_0(N)$

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where, under $\Gamma_0(N)$,

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f(\tau) = \prod_{l \mid N} \eta(l\tau)^{r_l}, \quad r_l \in \mathbb{Z}
$$

where

$$
\sum_{l \mid N} r_l = 4
$$
When $\Gamma_0(N)$ has $g = 1$:

- There is a function $x(\tau)$ of order 2, which can usually be constructed as an $\eta$ quotient.

- There is a cusp form $f(\tau)$ of weight 2, which can usually be constructed as an $\eta$ quotient.

- There is then necessarily an identity of the form

$$f(\tau)^2 \pi \text{id}_\tau = \frac{1}{\sqrt{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4}}$$

for constants $a_i$. 

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When $\mathbb{H}/\Gamma_0(N)$ has $g = 1$: 

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When \( \mathbb{H}/\Gamma_0(N) \) has \( g = 1 \):

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When $\mathbb{H}/\Gamma_0(N)$ has $g = 1$:

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Example: $\Gamma_0(15)$
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First,

$$x(\tau) = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3} = q - 3q^2 + 8q^4 - 9q^5 + \cdots$$

has simple poles at $\tau = 1/3$ and $\tau = 1/5$ and simple zeros at $\tau = 1/1$ and $\tau = 1/15$. 
Example: $\Gamma_0(15)$

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$$\eta_1 \eta_3 \eta_5 \eta_{15} = q - q^2 - q^3 - q^4 + q^5 + \cdots$$

is a cusp form.
Example: $\Gamma_0(15)$

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Next,

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is a cusp form.

Therefore,

$$\eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{dx}{\sqrt{a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0}}$$
Example: \( \Gamma_0(15) \)

First,

\[
x(\tau) = \frac{\eta_1^3 \eta_{15}^3}{\eta_3^3 \eta_5^3} = q - 3q^2 + 8q^4 - 9q^5 + \cdots
\]

has simple poles at \( \tau = 1/3 \) and \( \tau = 1/5 \) and simple zeros at \( \tau = 1/1 \) and \( \tau = 1/15 \).

Next,

\[
\eta_1 \eta_3 \eta_5 \eta_{15} = q - q^2 - q^3 - q^4 + q^5 + \cdots
\]

is a cusp form.

Therefore,

\[
\eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i \, d\tau = \frac{dx}{\sqrt{a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0}}
\]

\[
q = x + 3x^2 + 18x^3 + 127x^4 + O(x^5)
\]

\[
\left( \frac{1}{\eta_1 \eta_3 \eta_5 \eta_{15}} \frac{1}{2\pi i} \frac{dx}{d\tau} \right)^2 = 1 - 10x - 13x^2 + 10x^3 + x^4 + O(x^5)
\]

\[
= (x^2 - x - 1)(x^2 + 11x - 1)
\]
First, \( x(\tau) = \eta_4 \eta_6 \eta_2 \eta_1 \eta_2 \eta_2 = q - 2q^2 + 2q^4 + \cdots \) has simple poles at \( \tau = 1/3 \) and \( \tau = 1/8 \) and simple zeros at \( \tau = 1/1 \) and \( \tau = 1/24 \).

Next, \( \eta_2 \eta_4 \eta_6 \eta_12 \) is a cusp form.

Therefore, \( \eta_2 \eta_4 \eta_6 \eta_12 = dx \sqrt{x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0} = dx \sqrt{x^4 + 8x^3 + 2x^2 - 8x + 1} \)

\[
\int \tau \sqrt{3 \sqrt{2} - 1 \sqrt{2} + (\sqrt{2} - 1) \sqrt{3}} d\theta \sqrt{1 + 3 \sin^2 \theta},
\]
Example: $\Gamma_0(24)$

First,

$$x(\tau) = \frac{\eta_4 \eta_6 \eta_1^2 \eta_{24}^2}{\eta_2 \eta_{12} \eta_3^2 \eta_8^2} = q - 2q^2 + 2q^4 + \cdots$$

has simple poles at $\tau = 1/3$ and $\tau = 1/8$ and simple zeros at $\tau = 1/1$ and $\tau = 1/24$. 
Example: $\Gamma_0(24)$

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$$\int_{i\infty}^{\tau} \eta_2 \eta_4 \eta_6 \eta_{12} 2\pi i d\tau = \int_{\sin^{-1}\left(\frac{1}{\sqrt{3}} \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + \sqrt{3}}}\right)}^{\sin^{-1}\left(\frac{1}{\sqrt{3}} \sqrt{\frac{\sqrt{2} - 1 - (\sqrt{2} + \sqrt{3})x}{\sqrt{2} + \sqrt{3} + (\sqrt{2} - 1)x}\right)} \frac{d\theta}{\sqrt{1 + 3 \sin^2 \theta}},$$
Ramanujan has also given the example

\[ \int_{i \eta}^{\infty} \eta_{1} \eta_{35} \eta_{5} \eta_{7} 2 \pi i d \tau = \int_{0}^{x} x dx \sqrt{1 + x - x^2} \sqrt{1 - 5x - 9x^3 - 6x^5 - x^6} \]

where \( x = \eta_{1} \eta_{35} \eta_{5} \eta_{7} \)

which belong to \( \Gamma_{0}(35) \).

Such identities arise when \( g > 1 \) but \( H/\Gamma_{0}(N) \) still has a function of order 2.
Beyond Genus 1

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\[
\int_{i\infty}^{\tau} \eta_1 \eta_5 \eta_7 \eta_35 2\pi i d\tau = \int_{0}^{x} \frac{x \, dx}{\sqrt{1 + x - x^2} \sqrt{1 - 5x - 9x^3 - 6x^5 - x^6}}
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Such identities arise when \( g > 1 \) but \( \mathbb{H}/\Gamma_0(N) \) still has a function of order 2.
Beyond Genus 1

Theorem

Whenever there is a function $x(\tau)$ of order two on $\mathbb{H}/\Gamma_0(N)$, a basis of the space of holomorphic differentials for $\mathbb{H}/\Gamma_0(N)$ can be given as

$$\left\{ \frac{x^k dx}{\sqrt{\prod_{i=1}^{2g+1 \text{ or } 2g+2}(x - r_i)}} \right\}_{k=0}^{g-1}.$$ 

where $g$ is the genus of $\mathbb{H}/\Gamma_0(N)$. 

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\]

where \( g \) is the genus of \( \mathbb{H}/\Gamma_0(N) \).

The key to Ramanujan’s identities is:
We can still write down holomorphic differentials using cusp forms.
Beyond Genus 1: $\Gamma_0(30)$

There is a function of order 2 on $\mathbb{H}/\Gamma_0(30)$ given by

$$\eta_1 \eta_2 \eta_{15} \eta_{30} = 2 \pi i \text{id}_\tau = \frac{1}{2} \left( x - 1 \right) \frac{dx}{\sqrt{(x^2 - x - 1)(x^2 + 2x - 4)(x^4 - 3x^3 + 5x^2 - 6x + 4)}}.$$
\mathbb{H}/\Gamma_0(30) \text{ has genus 3.}
\( \mathbb{H}/\Gamma_0(30) \) has genus 3.

There is a function of order 2 on \( \mathbb{H}/\Gamma_0(30) \) given by

\[
x = 2 \frac{\eta_6 \eta_{10}}{\eta_1 \eta_{15}} = 2 + 2q + 4q^2 + 6q^3 + \cdots
\]
Beyond Genus 1: $\Gamma_0(30)$

$\mathbb{H}/\Gamma_0(30)$ has genus 3.

There is a function of order 2 on $\mathbb{H}/\Gamma_0(30)$ given by

$$x = 2 \frac{\eta_6 \eta_{10}}{\eta_1 \eta_{15}} = 2 + 2q + 4q^2 + 6q^3 + \cdots$$

$$\eta_1 \eta_3 \eta_5 \eta_{15} 2\pi i d\tau = \frac{2(x - 1)dx}{\sqrt{(x^2 - x - 1)(x^2 + 2x - 4)(x^4 - 3x^3 + 5x^2 - 6x + 4)}},$$

$$\eta_3 \eta_5 \eta_6 \eta_{10} 2\pi i d\tau = \frac{x(x - 1)dx}{\sqrt{(x^2 - x - 1)(x^2 + 2x - 4)(x^4 - 3x^3 + 5x^2 - 6x + 4)}},$$

$$\eta_1 \eta_2 \eta_{15} \eta_{30} 2\pi i d\tau = \frac{(x - 2)dx}{\sqrt{(x^2 - x - 1)(x^2 + 2x - 4)(x^4 - 3x^3 + 5x^2 - 6x + 4)}}.$$
Conclusion

We have interpreted Ramanujan's formulas in the light of complex function theory.

Ramanujan's methods would not have used these methods to prove his identities.

It is remarkable that he still discovered what one would call the isomorphism between the space of cusps forms of weight two and the space of holomorphic differentials.

If Ramanujan had worked a bit more, could he have discovered the previous three identities of level 30?
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Thank you!