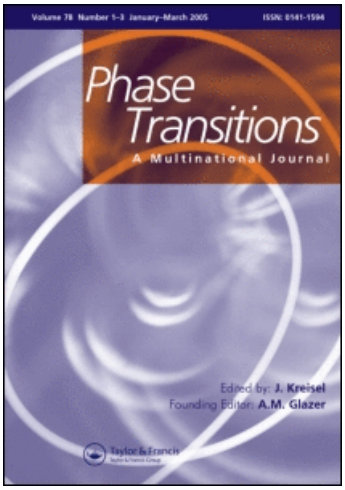

Phase Transitions
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713647403

Ferroic crystals and tensor distinction

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Online publication date: 09 September 2010

To cite this Article Litvin, Daniel B.(2010) 'Ferroic crystals and tensor distinction', Phase Transitions, 83: 9, 682 — 693
To link to this Article: DOI: 10.1080/01411594.2010.500217
URL: http://dx.doi.org/10.1080/01411594.2010.500217

Phase Transitions
Vol. 83, No. 9, September 2010, 682–693

Ferroic crystals and tensor distinction

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(*Received 11 April 2010; final version received 8 June 2010*)

The tensor distinction of ferroic crystals by physical property tensors is reviewed. We consider whether or not a physical property can simultaneously distinguish among all the domains which arise in a ferroic phase transition and between pairs of domains. Because of their central role in determining switching of domains, we also consider which components of a tensor distinguish between the pairs of domains. We also include the fourth primary ferroic physical property, the toroidal moment, in addition to magnetization, polarization, and strain.

Keywords: ferroic crystal; tensor distinction; ferroic phase transition

1. Introduction

A ferroic crystal contains two or more equally stable domains, volumes of the same homogeneous crystalline structure, in different spatial orientations. These domains can coexist in a crystal and may be distinguished by the values of components of certain macroscopic tensorial properties of the domains. Under a suitable driving force, the domain walls can be moved and the crystal can be switched from one orientation state to another [1–3]. Crystals in which domains may be distinguished by spontaneous polarization, spontaneous magnetization, and spontaneous strain are called primary ferroic crystals and named, respectively, ferroelectric, ferromagnetic, and ferroelastic crystals. Switching in these ferroic crystals is accomplished, respectively, by an electric field, a magnetic field, and a mechanical stress. A fourth type of primary ferroic crystal, a ferrotoroidic crystal, has been recently observed by Van Aken et al. [4]. Here the domains are distinguished by a toroidal moment [5–7]. For ferromagnetic/ferrotoroidic and antiferromagnetic/ferrotoroidic domains, depending on the symmetry, ferrotoroidic domains can be switched by crossed electric and magnetic fields, by collinear electric and magnetic fields, or by a magnetic field alone [8]. Crystals whose domains are characterized by differences in the dielectric permittivity tensor or piezoelectric tensor are examples of secondary ferroic crystals named as ferrobielectric and ferroelastoelectric crystals [3,9]. Switching is accomplished by an electric field in the former, and by a combination of an electric field and a mechanical stress in the latter.

Ferroic crystals have been discussed by Newnham [10,11] and Wadhawan [12,13], and secondary ferroic crystals in particular by Aizu [3], Newnham & Cross [14,15], and

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ISSN 0141–1594 print/ISSN 1029–0338 online

DOI: 10.1080/01411594.2010.500217
http://www.informaworld.com

Newnham & Skinner [16]. Listings of primary, secondary, and tertiary types of ferroics along with how switching is accomplished, including phenomenon involving the toroidal moment, are given by Schmid [8,17].

A brief review of magnetic groups is given in Section 2. In Section 3, physical property tensors are briefly discussed and an example is given of their use in determining possible switching of domains. As switching of domains depends on the differences in the physical properties, in Section 4 the physical property tensor distinction of domains is considered. This section is divided into three parts. First, global tensor distinction considers if a property tensor can or cannot distinguish among all domains that arise in a ferroic phase transition. Second, tensor pair distinction considers if a property tensor can distinguish between two domains, and third, tensor component distinction considers which specific components of a property tensor can distinguish between domains.

2. Magnetic groups

The analysis of tensor distinction of domains which arise in a phase transition is based on the change in the point group symmetry which arises in the transition. Because physical properties which could possibly distinguish the domains, as magnetization or toroidal moment, are not invariant under time inversion, it is crystallographic magnetic point groups which are used. Let $\boldsymbol{G}$ denote one of the 32 crystallographic point group types of a geometric crystal medium [18]. Magnetic point groups $\mathbf{M}$ are classified into 122 types $\boldsymbol{M}$ of magnetic point groups and subdivided into 32 reduced magnetic superfamilies each associated with one of the 32 crystallographic point group types $\boldsymbol{G}$. The *reduced magnetic superfamily of the crystallographic point group type* $\boldsymbol{G}$ consists of [18]:

(1) All groups $\mathbf{G}$ of type $\boldsymbol{G}$.
(2) All groups $\mathbf{G1}'$, where $\mathbf{G}$ is a group of type $\boldsymbol{G}$ and $\mathbf{1}'$ denotes the time inversion group consisting of the identity 1 and time inversion $1'$.
(3) All non-equivalent groups $\mathbf{G(D)} = \mathbf{D} + (\mathbf{G} - \mathbf{D})1'$, where $\mathbf{G}$ is a group of type $\boldsymbol{G}$ and $\mathbf{D}$ is a subgroup of index two of $\mathbf{G}$ of elements not coupled with time inversion.

For example, the magnetic superfamily of $\mathbf{G} = \mathbf{2/m}$ consists of:

$$\begin{aligned}
\text{(1) Groups } \mathbf{2/m} &= \{1, 2, \bar{1}, m\} \\
\text{(2) Groups } \mathbf{2/m1}' &= \{1, 2, \bar{1}, m, 1', 2', \bar{1}', m'\} \\
\text{(3) Groups } \mathbf{2/m(2)} &= \{1, 2, \bar{1}', m'\} = \mathbf{2/m}' \\
\mathbf{2/m(\bar{1})} &= \{1, 2', \bar{1}, m'\} = \mathbf{2}'\mathbf{/m}' \\
\mathbf{2/m(m)} &= \{1, 2', \bar{1}', m\} = \mathbf{2}'\mathbf{/m}
\end{aligned}$$

On the extreme right are the symbols of the groups of the third kind where the $\mathbf{G(D)}$ notation has been replaced with the *primed* notation which shows symmetries coupled with time inversion.

We note that a *magnetic superfamily*, as defined by Zamorzaev [19], does not include groups of type $\boldsymbol{G}$. The 122 types of crystallographic magnetic point groups divided into 32 reduced magnetic superfamilies are listed in Table 1. The 32 point group types $\boldsymbol{G}$ are listed in the left-hand column and the additional magnetic point group types $\boldsymbol{M}$ belonging to the reduced magnetic superfamily of each $\boldsymbol{G}$ are listed to the right of $\boldsymbol{G}$.

Table 1. The 32 reduced magnetic superfamilies of magnetic point groups.

1	**11′**			
$\bar{1}$	**$\bar{1}$1′**	**$\bar{1}$′**		
2	**21′**	**2′**		
m	**m1′**	**m′**		
2/m	**2/m1′**	**2′/m**	**2/m′**	**2′/m′**
222	**2221′**	**2′2′2**		
mm2	**mm21′**	**m′m2′**	**m′m′2**	
mmm	**mmm1′**	**m′mm**	**m′m′m**	**m′m′m′**
4	**41′**	**4′**		
$\bar{4}$	**$\bar{4}$1′**	**$\bar{4}$′**		
4/m	**4/m1′**	**4′/m**	**4/m′**	**4′/m′**
422	**4221′**	**4′22′**	**42′2′**	
4mm	**4mm1′**	**4′m′m**	**4m′m′**	
$\bar{4}$2m	**$\bar{4}$2m1′**	**$\bar{4}$′2′m**	**$\bar{4}$′2m′**	**$\bar{4}$2′m′**
4/mmm	**4/mmm1′**	**4/m′mm**	**4′/mm′m**	**4′/m′m′m**
	4/mm′m′	**4/m′m′m′**		
3	**31′**			
$\bar{3}$	**$\bar{3}$1′**	**$\bar{3}$′**		
32	**321′**	**32′**		
3m	**3m1′**	**3m′**		
$\bar{3}$m	**$\bar{3}$m1′**	**$\bar{3}$′m**	**$\bar{3}$′m′**	**$\bar{3}$m′**
6	**61′**	**6′**		
$\bar{6}$	**$\bar{6}$1′**	**$\bar{6}$′**		
6/m	**6/m1′**	**6′/m**	**6/m′**	**6′/m′**
622	**6221′**	**6′2′2**	**62′2′**	
6mm	**6mm1′**	**6′m′m**	**6m′m′**	
$\bar{6}$m2	**$\bar{6}$m21′**	**$\bar{6}$′m′2**	**$\bar{6}$′m2′**	**$\bar{6}$m′2′**
6/mmm	**6/mmm1′**	**6/m′mm**	**6′/mm′m**	**6′/m′m′m**
	6/mm′m′	**6/m′m′m′**		
23	**231′**			
m$\bar{3}$	**m$\bar{3}$1′**	**m′$\bar{3}$′**		
432	**4321′**	**4′32′**		
$\bar{4}$3m	**$\bar{4}$3m1′**	**$\bar{4}$′3m′**		
m$\bar{3}$m	**m$\bar{3}$m1′**	**m′$\bar{3}$′m**	**m$\bar{3}$m′**	**m′$\bar{3}$′m′**

3. Physical property tensors

The distinction of domains is made by differences in the physical properties of the domains. These physical properties are described by physical property tensors whose form is determined by the point group symmetry of the domains. Consequently, the derivation of physical property tensors invariant under point groups have been considered by many authors ([20–29] and references cited therein). Printed tables of a wide variety of physical property tensors have been given by Sirotin and Shaskolskaya [30] and Brandmuller et al. [31] and Volume D of the International Tables for Crystallography [32]. Computerized tabulations have been given by Litvin and Litvin [33], Popov et al. [34,35], and Ephraim et al. [36].

The form of the physical property tensors invariant under magnetic crystallographic point groups **M** can be derived from existing tables of physical property tensors invariant under point groups **G** of the 32 crystallographic group types ***G***: Let V denote a polar vector tensor and $V^n = V \times V \times \cdots \times V$ the nth ranked product of V, and let e and a denote zero-rank tensors that change sign under spatial inversion $\bar{1}$ and time

Table 2. The four primary ferroic property tensors, transformation properties, and Jahn symbols.

1	$\bar{1}$	$1'$	$\bar{1}'$	Property symbol	Jahn symbol	Name
1	1	1	1	ε_{ij}	$[V^2]$	Strain
1	−1	1	−1	P	V	Polarization
1	1	−1	−1	M	*ae*V	Magnetization
1	−1	−1	1	T	*a*V	Toroidal Moment

Note: Square bracket around a Jahn symbol denotes that the tensor is symmetrized.

inversion $1'$, respectively. Symbols V^n, eV^n, aV^n, and aeV^n denoting the transformational properties of a property tensor are known as the property tensor's *Jahn symbols* [20]. It has been shown [29,37] that the form of a physical property tensor invariant under a magnetic point group **M** is the same as the form of a physical property tensor transforming as a tensor V^n or eV^n invariant under a related group **G** of the reduced magnetic superfamily to which **M** belongs. Tables have been given listing the group **G** corresponding to the group **M** for all magnetic point groups and tensor types [37].

Table 2 lists the property tensors, strain, polarization, magnetization, and toroidal moment associated with the four primary ferroic crystal types, their transformation properties under the group $\bar{\mathbf{1}}\mathbf{1}' = \{1, \bar{1}, 1', \bar{1}'\}$ where $\bar{1}$ is spatial inversion and $1'$ is time inversion, and their Jahn symbol. Square brackets around a tensor symbol means the tensor is symmetrized. The forms of these four property tensors, invariant under a magnetic group **M** from each of the 122 types of magnetic point groups, are given in reference [38].

The switching of two domains is driven by a difference in their free enthalpy $\Delta g(\boldsymbol{E}, \boldsymbol{H}, \sigma, \boldsymbol{S})$, where $\boldsymbol{E}$, $\boldsymbol{H}$, and σ denote electric field, magnetic field, and stress, and $\boldsymbol{S} \sim (\boldsymbol{E} \times \boldsymbol{H})$ the source of the toroidal moment [17]

$$\begin{aligned}\Delta g(\boldsymbol{E}, \boldsymbol{H}, \sigma, \mathrm{S}) = {} & \Delta P_{(s)i}E_i + \Delta M_{(s)i}H_i + \Delta\varepsilon_{(s)ij}\sigma_{ij} + \Delta T_{(s)i}S_i + \tfrac{1}{2}\Delta\kappa_{ij}E_iE_j + \tfrac{1}{2}\Delta\chi_{ij}H_iH_j \\ & + \tfrac{1}{2}\Delta s_{ijkl}\sigma_{ij}\sigma_{kl} + \tfrac{1}{2}\Delta\tau_{ij}S_iS_j + \Delta d_{ijk}E_i\sigma_{jk} + \Delta q_{ijk}H_i\sigma_{jk} + \Delta\alpha_{ij}H_iE_j \\ & + \Delta\gamma_{ijk}S_i\sigma_{jk} + \Delta\varsigma_{ij}H_iS_j + \Delta\theta_{ij}E_iS_j + \text{higher order terms.}\end{aligned}$$

The name, corresponding ferroic type, and Jahn notation for each coefficient, i.e. physical property tensor, in the above expansion is given in Table 3.

As an example of how switching is accomplished [12], consider the form of the electric susceptibility tensor of a domain in a ferroic crystal invariant under the point group $\mathbf{2}_z$ [38]

$$\begin{pmatrix} \kappa_{xx} & \kappa_{xy} & 0 \\ \kappa_{xy} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}$$

and this tensor's form in a domain related to the first domain by the mirror plane m_x:

$$\begin{pmatrix} \kappa_{xx} & -\kappa_{xy} & 0 \\ -\kappa_{xy} & \kappa_{yy} & 0 \\ 0 & 0 & \kappa_{zz} \end{pmatrix}$$

Consequently $\Delta g = \ldots -\kappa_{xy}E_xE_y \ldots$ and the domains are ferrobielectric. As the magnetic susceptibility and toroidic susceptibility tensors have the same Jahn notation

Table 3. Physical property tensors: symbol, name, ferroic type, and Jahn symbol.

Primary ferroics			
$P_{(s)i}$	Spontaneous polarization	Ferroelectric	V
$M_{(s)i}$	Spontaneous magnetization	Ferromagnetic	aeV
$\varepsilon_{(s)ij}$	Spontaneous strain	Ferroelastic	[V^2]
$T_{(s)i}$	Spontaneous toroidal moment	Ferrotoroidic	aV
Secondary ferroics			
κ_{ij}	Electric susceptibility	Ferrobielectric	[V^2]
χ_{ij}	Magnetic susceptibility	Ferrobimagnetic	[V^2]
s_{ijkl}	Elastic compliance	Ferrobielastic	[[V^2]^2]
τ_{ij}	Toroidic susceptibility	Ferrobitoroidic	[V^2]
d_{ijk}	Piezoelectric coefficient	Ferroelastoelectric	V[V^2]
q_{ijk}	Piezomagnetic coefficient	Ferromagnetoelastic	aeV[V^2]
α_{ij}	Magnetoelectric coefficient	Ferromagnetoelectric	aeV^2
γ_{ijk}	Piezotoroidic coefficient	Ferroelastotoroidic	aV[V^2]
ζ_{ij}	Magnetotoroidic coefficient	Ferromagnetotoroidic	eV^2
θ_{ij}	Electrotoroidic coefficent	Ferroelectrotoroidic	aV^2

as the electric susceptibility tensor (Table 3), their transformational properties are the same. Consequently, in the above example, their tensor forms are the same as that of the electric susceptibility tensor, and the domains are also ferrobimagnetic and ferrobitoroidic.

4. Tensor distinction of domains

Consider a ferroic phase transition, that is, a phase transition of a crystalline structure from a phase of higher point group symmetry **C** to a phase of lower point group symmetry **J**. In the lower symmetry phase there are $n = |\mathbf{C}|/|\mathbf{J}|$ *single domain states* $S_1, S_2, \ldots, S_n$, where $|\mathbf{C}|$ and $|\mathbf{J}|$ denote the number of elements in **C** and **J**, respectively. Single domain states have the same crystalline structure and differ only in their orientation in space. *Domain states* will refer to the bulk structures, with their specific orientations in space, of domains in a polydomain sample. Several disconnected domains can have the same domain state. Domain states represent then the structures that appear in a polydomain sample, irrespective of which domain they are in.

In *non-ferroelastic* polydomain phases, the orientation of each domain state coincides with the orientation of a single domain state. The number of domain states is therefore the same as the number of single domain states. In *ferroelastic* polydomain phases, because of disorientations, that is, rotations of domains that arise as a result of the requirement that neighboring domains in the polydomain sample must meet along a coherent boundary, domain states in general differ in their orientation from single domain states. The number of domain states is then, in general, greater than the number of single domain states. In distinction to domains in non-ferroelastic polydomain phases, the orientations are then, in general, not related by elements of **C**. We shall consider here the ferroelastic polydomain phases in the *parent-clamping approximation* [39,40] which disregards the disorientations. We disregard these disorientations; the number and orientation of domain states in a ferroelastic polydomain sample then also coincide with the number and orientations of single domain states, as in non-ferroelastic polydomain samples.

The domain states, also denoted by $S_1, S_2, \ldots, S_n$, are related by elements of **C** not in **J**: we subdivide the group **C** into a left *coset* decomposition with respect to **J**

$$\mathbf{C} = \mathbf{J} + c_2\mathbf{J} + c_3\mathbf{J} + \cdots + c_n\mathbf{J} \tag{1}$$

where the elements c_i, $i = 1, 2, \ldots, n$, $c_1 = 1$, are called the *coset representatives* of the coset decomposition of **C** with respect to **J**. Denoting the elements of the group **J** by $\{1, j_2, \ldots, j_s\}$, each coset $c_i\mathbf{J}$ denotes a subset of elements $\{c_i, c_i j_2, \ldots, c_i j_s\}$ of **C**. The choice of coset representatives is not unique because the coset representative c_i can be replaced by $c_i j$, where j is any element of **J**. Defining the domain state S_1 as the domain invariant under **J**, the orientations of the remaining domain states are related to S_1 by the coset representatives:

$$S_i = c_i S_1, \quad i = 2, 3, \ldots, n. \tag{2}$$

In addition, each domain state S_i, $i = 1, 2, \ldots, n$, is invariant under the group $\mathbf{J}_i = c_i \mathbf{J} c_i^{-1}$.

For example, if $\mathbf{C} = 4_z/\mathbf{m}_z\mathbf{m}'_x\mathbf{m}'_{xy}$ and $\mathbf{J} = 2'_{xy}/\mathbf{m}'_{xy}$, then $\mathbf{C} = \mathbf{J} + c_2\mathbf{J} + c_3\mathbf{J} + c_4\mathbf{J}$ and:

$$\begin{aligned}
\mathbf{J} &= \left\{1, \bar{1}, 2'_{xy}, m'_{xy}\right\} & c_1 &\equiv 1 & \mathbf{J}_1 &= c_1\mathbf{J}c_1^{-1} = \mathbf{2}'_{xy}/\mathbf{m}'_{xy} \\
c_2\mathbf{J} &= \left\{2_z, m_z, 2'_{\bar{x}y}, m'_{\bar{x}y}\right\} & c_2 &= 2_z \;\; S_2 = 2_z S_1 & \mathbf{J}_2 &= c_2\mathbf{J}c_2^{-1} = \mathbf{2}'_{xy}/\mathbf{m}'_{xy} \\
c_3\mathbf{J} &= \left\{2'_y, 4_z, 4_z, m'_y\right\} & c_3 &= 2'_y \;\; S_3 = 2'_y S_1 & \mathbf{J}_3 &= c_3\mathbf{J}c_3^{-1} = \mathbf{2}'_{\bar{x}y}/\mathbf{m}'_{\bar{x}y} \\
c_4\mathbf{J} &= \{2'_x, 4^3_z, m'_x, \bar{4}^3_z\} & c_4 &= 2'_x \;\; S_4 = 2'_x S_1 & \mathbf{J}_4 &= c_4\mathbf{J}c_4^{-1} = \mathbf{2}'_{\bar{x}y}/\mathbf{m}'_{\bar{x}y}
\end{aligned}$$

We shall be interested here in what is referred to as *tensor distinction*, i.e. the distinction of the domains in a polydomain phase of a ferroic phase transition by the macroscopic tensorial physical properties of tensor types $\boldsymbol{T}$. As the set of domain states represents the structure of all domains in a polydomain phase, we consider the tensor distinction of the domain states.

We denote by T_i, $i = 1, 2, \ldots, n$, the tensors of a tensor type $\boldsymbol{T}$ in the set of domain states $S_1, S_2, \ldots, S_n$ of a polydomain sample, and are related by the coset representatives in Equation (1):

$$T_i = c_i T_1, \quad i = 1, 2, \ldots, n \tag{3}$$

where $c_i T_1$ represents the transformation of the tensor form T_1 under the operation c_i [21]. The tensors are all given in a single coordinate system, e.g. the coordinate system of the parent phase structure or of one of the domain states. A tensor type $\boldsymbol{T}$ is said to be able to distinguish between two domain states, with corresponding tensors T_i and T_j of the type $\boldsymbol{T}$, if $T_i \neq T_j$. In particular, we consider three types of tensor distinction problems separately:

(a) *Global tensor distinction*: We consider whether or not a tensor of type $\boldsymbol{T}$ can distinguish among all domain states, i.e., if $T_i \neq T_j$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$.
(b) *Domain pair tensor distinction*: For a pair of domain states, S_i and S_j, $i \neq j$, we consider whether or not a tensor of type $\boldsymbol{T}$ can distinguish between the domain states, i.e., if $T_i = T_j$ or $T_i \neq T_j$.
(c) *Tensor component distinction*: We consider which components of a tensor of type $\boldsymbol{T}$ can distinguish between domain states and domain pair.

4.1. *Global tensor distinction*

We consider here whether or not tensors T_i, $i=1,2,\ldots,n$, of the type $\boldsymbol{T}$ can distinguish among all domain states S_i, $i=1,2,\ldots,n$, which arise in a ferroic phase transition. Following the terminology of Aizu [1], if the set of tensors T_i, $i=1,2,\ldots,n$, are all distinct, then we say that the tensor type $\boldsymbol{T}$ provides a *full* distinction of the domain states S_i, $i=1,2,\ldots,n$. If the set of tensors T_i, $i=1,2,\ldots,n$, are not all distinct but also not all identical, then we say that the tensor type $\boldsymbol{T}$ provides a *partial* distinction. If the tensors are all identical, then the terminology is that a type $\boldsymbol{T}$ tensor provides a *null* distinction. Litvin [41] subdivided the "null" category into two: if the tensors are all identical and non-zero the terminology remains "null distinction." If the tensors are all identical and zero then the terminology becomes "zero distinction." In tabulations full, partial, null, and zero distinction are denoted, respectively, by the letters "F", "P", "N", and "Z".

We shall use the Aizu symbol **C***F***J** to characterize a ferroic phase transition consisting of the point group **C** of the high symmetry phase, the letter F to denote a ferroic phase transition, and the point group **J** of the low symmetry phase. While **C***F***J** will be used as a general symbol for such a phase transition, we shall categorize these symbols into three kinds **G***F***Sub**, **G1**′*F***Sub**, and **G(D)***F***Sub** to correspond to the three kinds of magnetic point groups of the high symmetry phase. The symbol **Sub** represents a subgroup of the point group on the left-hand-side of the symbol F.

A method of determining the global tensor distinction was given by Litvin [41] and the global tensor distinction of classes of ferroic phase transitions **G***F***Sub** for all tensor types $\boldsymbol{T}$ of rank $n \leq 4$ is given in [42]. Aizu [1] has given the global tensor distinction of classes of ferroic phase transitions **G1**′*F***Sub** for magnetization, polarization, and strain. This was extended by Litvin [43] to include global distinction of classes **G1**′*F***Sub** by toroidal moment. In addition, in [43] there was given inverse tables to look up which ferroic phase transitions **G1**′*F***Sub** have particular kinds of distinctions by these four tensor types, and also the relationship to Schmid's concept [44] of ensembles of species. The global distinction *via* toroidal moment, magnetization, polarization, and strain tensor types of ferroic phase transitions has been extended from the 773 classes of transitions **G1**′*F***Sub** to all classes **C***F***J**, i.e. to the 212 classes **G***F***Sub** and to the 616 classes **G(D)***F***Sub** [45]. Examples of this global tensor distinction are given in Table 4.

4.2. *Domain pair tensor distinction*

In a ferroic phase transition **C***F***J** we consider the tensor distinction, with the tensors T_i and T_j, of ordered domain pairs $\{S_i, S_j\}$ $i \neq j$, made up of domains defined in Equation (2). The tensors T_i and T_j can be determined as follows: T_i is the form of the tensor type $\boldsymbol{T}$ invariant under $\mathbf{J}_i$ and T_j can be determined from T_i *via* $T_j = c_{ij}\, T_i = c_j c_c^{-1} T_i$, where c_{ij} is an operation of C which transforms the domain state S_i into the domain state S_j, i.e. $S_j = c_{ij} S_i$. Consequently, in a ferroic phase transition **C***F***J** the tensor distinction of a domain pair $\{S_i, S_j\}$ is determined by the point group $\mathbf{J}_i$ and the operation of the set of elements $c_{ij}\mathbf{J}_i$ of **C**. There is a one-to-one correspondence between the domain pair $\{S_i, S_j\}$ and the sets $\{\mathbf{J}_i, c_{ij}\mathbf{J}_i\}$ which determine the domain pair tensor distinction.

Domain pairs $\{S_i, S_j\}$ can be classified into classes in a manner where all domain pairs in the same class have the same domain pair tensor distinction [46]. In determining all possible domain pair tensor distinction in a ferroic phase transition **C***F***J** we consequently need to consider only one set $\{\mathbf{J}_1, c_j\mathbf{J}_1\}$ corresponding to one domain pair from each class of domain pairs. For example, consider the ferroic phase transition

Table 4. Examples of global distinction for the four property tensor toroidal moment (T), magnetization (M), polarization (P), and strain (ε_{ij}).

	T	M	P	ε_{ij}
GF Subgroup				
$\mathbf{m}_x\mathbf{m}_y\mathbf{m}_z F\mathbf{1}$	F	P	F	P
$\mathbf{m}_x\mathbf{m}_y\mathbf{m}_z F\mathbf{1}$-	Z	F	Z	F
$\mathbf{m}_x\mathbf{m}_y\mathbf{m}_z F\mathbf{2}_z$	P	P	P	P
$\mathbf{m}_x\mathbf{m}_y\mathbf{m}_z F\mathbf{m}_z$	F	P	F	P
$\mathbf{m}_x\mathbf{m}_y\mathbf{m}_z F\mathbf{2}_z/\mathbf{m}_z$	Z	F	Z	F
$\mathbf{m}_x\mathbf{m}_y\mathbf{m}_z F\mathbf{2}_x\mathbf{2}_y\mathbf{2}_z$	Z	Z	Z	N
$\mathbf{m}_x\mathbf{m}_y\mathbf{m}_z F\mathbf{m}_x\mathbf{m}_y\mathbf{2}_z$	F	Z	F	N
G$\mathbf{1}'F$ Subgroup				
$\mathbf{4}_z\mathbf{1}'F\mathbf{1}$	F	F	F	F
$\mathbf{4}_z\mathbf{1}'F\mathbf{1}'$	Z	Z	F	F
$\mathbf{4}_z\mathbf{1}'F\mathbf{2}_z$	P	P	N	F
$4_z\mathbf{1}'F\mathbf{2}'_z$	F	F	N	F
$\mathbf{4}_z\mathbf{1}'F\mathbf{2}_z\mathbf{1}'$	Z	Z	N	F
$\mathbf{4}_z\mathbf{1}'F4_z$	F	F	N	N
$4_z\mathbf{1}'F4'_z$	Z	Z	N	N
G(D)F Subgroup				
$\mathbf{4}_z/\mathbf{m}'_z F\mathbf{1}$	P	F	F	P
$\mathbf{4}_z/\mathbf{m}'_z F\mathbf{1}$-$'$	F	Z	Z	F
$\mathbf{4}_z/\mathbf{m}'_z F\mathbf{2}_z$	N	P	P	P
$\mathbf{4}_z/\mathbf{m}'_z F\mathbf{m}'_z$	N	F	F	P
$\mathbf{4}_z/\mathbf{m}'_z F\mathbf{2}_z/\mathbf{m}'_z$	N	Z	Z	F
$\mathbf{4}_z/\mathbf{m}'_z F\mathbf{4}_z$	N	F	F	N
$\mathbf{4}_z/\mathbf{m}'_z F\mathbf{4}'_z$	N	Z	Z	N

Note: "F" denotes full distinction, "P" partial distinction, "N" null distinction, and "Z" zero distinction.

Table 5. The domain pair tensor distinction for the physical properties toroidal moment (T), magnetization (M), polarization (P), and strain (ε_{ij}) for the ferroic phase transition $\mathbf{C}F\mathbf{J} = \mathbf{m}'_z\bar{3}'_{xyz}\mathbf{m}'_{xy}F\mathbf{m}'_{xy}\mathbf{m}'_z 2_{\bar{x}y}$.

$\mathbf{J}_1$	$c_j\mathbf{J}_1$				T	M	P	ε_{ij}
$\mathbf{m}'_{xy}\mathbf{m}'_z\mathbf{2}_{\bar{x}y}$	$2'_{\bar{y}z}$	$3_{x\bar{y}z}$	$\bar{4}'_x$	$3^{-5}_{x\bar{y}z}{}'$	Z	F	F	F
$\mathbf{m}'_{xy}\mathbf{m}'_z\mathbf{2}_{\bar{x}y}$	$\bar{1}'$	2_{xy}	$\mathbf{m}'_{\bar{x}y}$	2_z	Z	F	F	N
$\mathbf{m}'_{xy}\mathbf{m}'_z\mathbf{2}_{\bar{x}y}$	$\mathbf{m}'_{\bar{y}z}$	$\bar{3}'_{\bar{x}yz}$	4_x	3^2_{xyz}	Z	F	F	F
$\mathbf{m}'_{xy}\mathbf{m}'_z\mathbf{2}_{\bar{x}y}$	2_x	4^{-3}_z	$\mathbf{m}'_y$	$4'_z$	Z	F	F	F

Note: "F" denotes full distinction, "N" null distinction, and "Z" zero distinction.

$\mathbf{C}F\mathbf{J} = \mathbf{m}'_z\bar{3}'_{xyz}\mathbf{m}'_{xy}F\mathbf{m}'_{xy}\mathbf{m}'_z\mathbf{2}_{\bar{x}y}$, where there arises 12 domain states S_i, $i = 1, 2, \ldots, 12$. The 132 domain pairs $\{S_i, S_j\}$ $i \neq j$ are classified into four classes of domain pairs. The domain pair tensor distinction of these four classes with respect to the tensors toroidal moment, magnetization, polarization, and strain are given in Table 5. For a complete listing of the tensor pair distinction for these four physical properties and all ferroic phase transitions see reference [45].

Table 6. Examples of component distinction of magnetic nonferroelastic domain pairs $\{S_1, S_j\}$ for the physical property tensor of the type eV^2.

J	c_j**J**	n/m
$\mathbf{m}'_x\mathbf{m}'_y\mathbf{2}_z$	$1'm_xm_y2'_z$	0/2
	$T_1 = 0,\ T_{xy},\ 0$ $T_{yx},\ 0,\ 0$ $0,\ 0,\ 0$	$T_j = 0,\ T_{xy},\ 0$ $T_{yx},\ 0,\ 0$ $0,\ 0,\ 0$
$\mathbf{m}'_x\mathbf{m}'_y\mathbf{2}_z$	$\bar{1}'2_x2_ym'_z$	2/0
	$T_1 = 0,\ T_{xy},\ 0$ $T_{yx},\ 0,\ 0$ $0,\ 0,\ 0$	$T_j = 0,\ -T_{xy},\ 0$ $-T_{yx},\ 0,\ 0$ $0,\ 0,\ 0$
$\mathbf{4}_z$	$2'_x2'_y2'_{xy}2'_{\bar{x}y}$	1/2
	$T_1 = T_{xx},\ T_{xy},\ 0$ $-T_{xy},\ T_{xx},\ 0$ $0,\ 0,\ T_{zz}$	$T_j = T_{xx},\ -T_{xy},\ 0$ $T_{xy},\ T_{xx},\ 0$ $0,\ 0,\ T_{zz}$
$\mathbf{4}_z$	$m_xm_ym_{xy}m_{\bar{x}y}$	2/1
	$T_1 = T_{xx},\ T_{xy},\ 0$ $-T_{xy},\ T_{xx},\ 0$ $0,\ 0,\ T_{zz}$	$T_j = -T_{xx},\ T_{xy},\ 0$ $-T_{xy},\ -T_{xx},\ 0$ $0,\ 0,\ -T_{zz}$

Notes: **J** is the point group of domain S_1 and c_jJ are elements which transform T_1 into T_j. "n" is the number of independent components which change sign, and "m" the number which stay constant.

4.3. *Tensor component distinction*

To determine the tensor components which distinguish all domains in a ferroic phase transition **C***F***J** one needs to know the set of tensor forms $\{T_1, c_2T_1, \ldots, c_nT_1\}$, where T_1 is the form of the tensor type ***T*** invariant under **J** and $c_2, c_3, \ldots c_n$ are a set of coset representatives, see Equation (1), of the coset decomposition of **C** with respect to **J**. The sets of tensor forms $\{T_1, c_2T_1, \ldots, c_nT_1\}$ for the physical properties magnetization and polarization, for the 656 groups **C** which are subgroups of $6_z/m_zm_xm_11'$ and $m_z\bar{3}_{xyz}m_{xy}1'$, and all possible groups **J**, have been tabulated [47].

For the switching of domains which arise in a ferroic phase transition **C***F***J** one considers the tensor component distinction of domain pairs$\{S_i, S_j\}$ $i \neq j$: One first classifies all domain pairs in the transition in the same manner as in the domain pair tensor distinction in the previous section. All domain pairs in the same class are distinguished by

the same tensor components. For one domain pair $\{S_1, S_j\}$ $1 \neq j$ in each class one then determines the tensor T_1, invariant under **J**, and $T_j = c_j T_1$. For the 141 classes of transposable non-ferroelastic magneto-electric transitions, such a tensor component distinction has been given for the 12 physical property tensor in [48], for the 212 classes of piezoelectric non-ferromagnetic domain pairs, the tensor component distinction has been given for nine physical property tensor types [49], and for the 309 classes of magnetic non-ferroelastic domain pairs [50] the tensor component distinction has been determined for a wide variety of physical property tensor types [51]. Examples from the last set are given in Table 6.

Acknowledgements

D.B. Litvin would like to express his appreciation and thanks to Dr Wadhawan for the personal introduction, many many years ago, to the realm of domains and ferroic phase transitions.

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