ONE-DIMENSIONAL QUASI-CRYSTALS AND SEQUENCES OF ONES AND ZEROS

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We derive the relationship between one-dimensional quasi-crystals and sequences of ones and zeros introduced by de Bruijn. Properties of these sequences are used to derive explicit expressions for the positions of the quasi-crystal atoms, for nearest neighbor distances, and a classification in which all identical but shifted quasi-crystals are classified into a single class.

In an alloy of Al and Mn, Shechtman, Blech, Gratias and Cahn [1] discovered a crystal-like structure with an icosahedral diffraction pattern. Levine and Steinhardt [2] soon after introduced a model structure which gives rise to a similar diffraction pattern. Geometric projection methods of constructing such crystal-like structures, called quasi-crystals, have been given by Kramer and Neri [3], Elser [4], Zia and Dallas [5], and Duneau and Katz [6].

Geometric projection methods to construct one-dimensional quasi-crystals consider a two-dimensional square lattice of points with sides of the square of unit length. In addition to an orthogonal X, Y coordinate system, see fig. 1, a second rotated X', Y' coordinate system is introduced. The angle of rotation \( \theta \) is such that \( \tan \theta \) is irrational. In the "cell" method of constructing one-dimensional quasi-crystals [4], a line A, see fig. 1, is drawn parallel to the X' axis and displaced a distance \( d \) along the Y' axis. The line is assumed not to intersect any lattice point. The lower left-hand corner of each square cut by this line is projected onto line A. The array of projected points constitutes a one-dimensional quasi-crystal with two

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Fig. 1. Circles on line A represent atoms of a one-dimensional quasi-crystal. Hash marks on the X-axis represent the projections of the intersections of line A with the horizontal lattice lines.

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basic lengths, of lengths \( \cos \theta \) and \( \sin \theta \).

De Bruijn [7] has defined the following sequences of ones and zeros:

\[
P_\gamma(z) = \lfloor \gamma + (z + 1)/\alpha \rfloor - \lfloor \gamma + z/\alpha \rfloor, \tag{1}
\]

\[
Q_\gamma(z) = \lfloor \gamma + (z + 1)/\alpha \rfloor - \lfloor \gamma + z/\alpha \rfloor, \tag{2}
\]

where \( \gamma \) and \( \alpha > 1 \) are real numbers. \( \lfloor x \rfloor \), called the floor of \( x \), is the largest integer less than or equal to \( x \), i.e. the integer part of \( x \). \( \lceil x \rceil \), called the roof of \( x \) is the smallest integer greater than or equal to \( x \). The sequence \( P_\gamma \) takes its ones on the set

\[
\{ \lfloor \alpha(n - \gamma) \rfloor | n \in \mathbb{Z} \}, \tag{3a}
\]

and its zeros on the set

\[
\{ \lfloor (n + \gamma)\alpha/(\alpha - 1) \rfloor - 1 | n \in \mathbb{Z} \}. \tag{3b}
\]

The sequence \( Q_\gamma \) takes its ones on the set

\[
\{ \lfloor \alpha(n - \gamma) \rfloor - 1 | n \in \mathbb{Z} \}, \tag{4a}
\]

and its zeros on the set

\[
\{ \lfloor (n + \gamma)\alpha/(\alpha - 1) \rfloor | n \in \mathbb{Z} \}. \tag{4b}
\]

One-dimensional quasi-crystals are related to these sequences via the following two theorems:

**Theorem 1.** A mapping exists between the sequences of ones and zeros of \( P_\gamma(z) \) with

\[
\gamma = -d/\sin \theta \tag{5a}
\]

and

\[
\alpha = \tan \theta, \tag{5b}
\]

and the one-dimensional quasi-crystal constructed with a line \( A \), see fig. 1, rotated by an angle \( \theta \) and displaced a distance \( d \). To each zero of \( P_\gamma(z) \) there corresponds a segment of length \( \sin \theta \), and to each one there correspond two segments, one of length \( \sin \theta \) followed by one of length \( \cos \theta \).

**Proof.** Let \( x_\gamma(z) \) denote the projection of the intersection of line \( A \) with a lattice line at \( Y = z \) onto the \( X \) axis. \( x_\gamma(z) = \gamma + z/\alpha \) where \( \gamma \) and \( \alpha \) are defined in eqs. (5a) and (5b). To a pair of consecutive points \( x_\gamma(z) \) and \( x_\gamma(z + 1) \) which fall between the same vertical lattice lines there corresponds a segment of length \( \sin \theta \). To a pair which fall on opposite sides of a vertical lattice line there correspond two consecutive segments of length \( \sin \theta \) and \( \cos \theta \). For example, see fig. 1, to the pair of points \( x_\gamma(0) = \gamma \) and \( x_\gamma(1) = \gamma + 1/\alpha \) corresponds the segment BC, to the pair of points \( x_\gamma(1) = \gamma + 1/\alpha \) and \( x_\gamma(2) = \gamma + 2/\alpha \), the segments CD and DE. Finally, from eq. (1), the value of \( P_\gamma(z) \) is zero if \( x_\gamma(z) \) and \( x_\gamma(z + 1) \) fall between the same vertical lattice lines, and is one if on opposite sides of a vertical lattice line.

A second relationship gives a one-to-one correspondence between a sequence of ones and zeros and the sequence of segments of lengths \( \sin \theta \) and \( \cos \theta \) of the one-dimensional quasi-crystal:

**Theorem 2.** The sequence of ones and zeros \( Q_\gamma(z) \), where

\[
Q_\gamma(z) = \lfloor \gamma^* + (z + 1)/\alpha^* \rfloor - \lfloor \gamma^* + z/\alpha^* \rfloor \tag{6}
\]

and

\[
\alpha^* = 1 + \alpha^{-1}, \tag{7a}
\]

\[
-\alpha^*\gamma^* = \gamma + \lceil \gamma \rceil + 1, \tag{7b}
\]

with \( \gamma \) and \( \alpha \) defined in eqs. (5a) and (5b), is in a one-to-one correspondence with the sequence of segments of a one-dimensional quasi-crystal: Each one corresponds to a segment of length \( \sin \theta \), and each zero to a segment of length \( \cos \theta \).

**Proof.** We transform the sequence \( P_\gamma(z) \) of theorem 1 by replacing each zero by one and each one by \( \alpha \), i.e. \( 0 \rightarrow 1 \) and \( 1 \rightarrow \alpha \). In this new sequence, each one corresponds to a segment of the one-dimensional quasi-crystal of length \( \sin \theta \) and each zero to a segment of length \( \cos \theta \). In this new sequence we indicate the places of the ones: To \( P_\gamma(n) \) there corresponds a group 1 or \( 0 \) starting at index \( M \). \( M \) is twice the number of ones among \( P_\gamma(0), P_\gamma(1), \ldots, P_\gamma(n - 1) \), plus the number of zeros among \( P_\gamma(0), P_\gamma(1), \ldots, P_\gamma(n - 1) \). Using eq. (1), we have \( M = \lfloor \gamma + n/\alpha \rfloor - \lfloor \gamma \rfloor + n \). Consequently, the new sequence has ones at

\[
\{ \lfloor \gamma + n/\alpha \rfloor - \lfloor \gamma \rfloor + n | n \in \mathbb{Z} \},
\]
which can be written as

\[ \{ [\alpha^*(n - \gamma^*)] - 1 | n \in \mathbb{Z} \}, \]  

(8)

with \( \alpha^* \) and \( \gamma^* \) defined by eqs. (7a) and (7b). On comparing eq. (8) with eq. (4a) we have that the new sequence is given by (2) with \( \gamma \) and \( \alpha \) replaced by \( \gamma^* \) and \( \alpha^* \).

An algebraic expression for the coordinates of the one-dimensional quasi-crystal atoms follows from theorem 2:

**Theorem 3.** The positions \( x' \) of the atoms of a one-dimensional quasi-crystal are given by

\[ x'(z) = z \cos \theta + ([\gamma^* + z/\alpha^*] - [\gamma^*]) \times (\sin \theta - \cos \theta) \]  

(9)

for all \( z \in \mathbb{Z} \).

**Proof.** We choose the origin at the atom, indexed by \( z = 0 \), on the left of the segment corresponding to \( Q_0^*(0) \). The position \( x'(z) \) of the \( z \)th atom is at the right of the segment corresponding to \( Q_{z-1}^*(z-1) \). \( x'(z) \) is equal to \( z \) times the number of ones among \( Q_{-1}^*(0), Q_{-1}^*(1), \ldots, Q_{-1}^*(z-1) \), plus \( \cos \theta \) times the number of zeros among \( Q_0^*(0), Q_1^*(1), \ldots, Q_{z-1}^*(z-1) \). Using eq. (6), theorem 3 follows.

From theorem 3 we have that the nearest neighbor distances of the \( z \)th quasi-crystal atom are given by

\[ \cos \theta + Q_{z-1}^*(N)(\sin \theta - \cos \theta) \]

for \( N = z \) and \( z + 1 \).

Sequences \( P_\gamma(z) \) (and \( Q_\gamma(z) \)) can be classified into equivalence classes: We define two sequences \( P_\gamma(z) \) and \( P_\gamma(z) \) to belong to the same class, and said to be equivalent, if for all \( z \) and integer \( N \):

\[ P_\gamma(z) = P_\gamma(z + N). \]  

(10)

Two equivalent sequences are then identical but shifted sequences of ones and zeros. The following theorem holds for both sequences \( P_\gamma(z) \) and \( Q_\gamma(z) \):

**Theorem 4.** Sequences \( P_{\gamma+M+N/\alpha}(z) \) and \( P_{\gamma}(z) \), where \( M \) and \( N \) are integers, are equivalent.

The proof follows by substituting \( \gamma + M + N/\alpha \) for \( \gamma \) into eq. (1)

\[ P_{\gamma+M+N/\alpha}(z) = \left\lfloor \gamma + M + N/\alpha + (z + 1)/\alpha \right\rfloor \]

\[ = \left\lfloor \gamma + (z + N + 1)/\alpha \right\rfloor \]

\[ = \left\lfloor \gamma + (z + N)/\alpha \right\rfloor \]

\[ = P_{\gamma}(z + N), \]

and using the definition of equivalent sequences, eq. (10).

Two one-dimensional quasi-crystals are said to be equivalent if the corresponding sequences \( P_\gamma(z) \) (and \( Q_\gamma(z) \)) are equivalent. Two equivalent one-dimensional quasi-crystals are identical but shifted sequences of lengths \( \sin \theta \) and \( \cos \theta \). In terms of the parameters \( \theta \) and \( d \) of the projection method of constructing one-dimensional quasi-crystals we have:

**Theorem 5.** One-dimensional quasi-crystals constructed with angle \( \theta \) and displacements \( d + M \cos \theta + N \sin \theta \), where \( M \) and \( N \) are arbitrary integers, are equivalent one-dimensional quasi-crystals.

**Proof.** Let \( P_\gamma(z) \) denote the sequence which by theorem 1 corresponds to the one-dimensional quasi-crystal constructed with the angle \( \theta \) and displacement \( d \). From eqs. (5a) and (5b) it follows that corresponding to the one-dimensional quasi-crystals constructed with angle \( \theta \) and displacements \( d + M \cos \theta + N \sin \theta \) are the sequences \( P_{\gamma+M+N/\alpha}(z) \). Since by theorem 4 these sequences are equivalent, it follows that the corresponding one-dimensional quasi-crystals are also equivalent.

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**References**


