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TENSORIAL CLASSIFICATION OF DOMAIN PAIRS

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A classification of domain pairs via a group theoretical classification of the corresponding pairs of a *full* physical property tensor characterizing the domains has been introduced by V. Janovec (Czech. J. Phys., **B22**, 974 (1972)). This classification scheme is extended here from the case of *full* physical property tensors, where each domain is characterized by a unique form of the physical property tensor, to the case of *partial* physical property tensors, where more than a single domain is characterized by a specific form of the physical property tensor.

I. INTRODUCTION

A ferroic crystal contains two or more equally stable domains of the same structure but of different spatial orientation. These domain can coexist in a crystal and may be distinguished by the values of components of certain macroscopic tensorial physical properties of the domains. Crystals in which the domains may be distinguished by spontaneous polarization, magnetization, or strain are called primary ferroic crystals. Crystals whose domains are characterized by differences in the dielectric permittivity tensor or piezoelectric tensor are examples of secondary ferroic crystals.[1,2] Ferroic crystals have been discussed by Newnham[3] and Wadhawan[4] and secondary ferroic crystals in particular by Aizu,[1] Newnham and Cross[5,6] and Newnham and Skinner.[7]

Aizu[8–10] has introduced point group classification schemes for ferroic crystals. These classification schemes are based on relationships between the point group symmetries of the domains of a ferroic crystal and the point group symmetry of the non-ferroic or *prototypic* high symmetry phase of the crystal. In addition, each class of ferroic crystals has been given a tensorial classification according to a macroscopic tensorial physical property tensor's ability to distinguish between the domains. Aizu has tabulated the tensorial classification of primary ferroic crystals.[8,11] A method to determine the tensorial classification of ferroic crystals with respect to an arbitrary macroscopic tensorial physical property tensor has been given by Litvin[12] and used to determine the tensorial classification of non-magnetic crystals for all physical property tensors of rank less than or equal to four.[13]

In the study of the mutual relationships between domains, the simplest object one can consider is a pair of domains, i.e., a domain pair. A classification of domain pairs via a tensorial classification of corresponding pairs of a *full* physical property

*Present Address: General Electric Aerospace, P.O. Box 8048, Philadelphia, PA 19101 tensor characterizing the domains was introduced by Janovec.[14] This classification scheme is extended here from the case of *full* physical property tensors, where each domain is characterized by a unique form of the physical property tensor, to the general case of *partial* physical property tensors, where more than a single domain is characterized by the same specific form of the physical property tensor.

In Section II we review the definitions of the subgroups of the high symmetry phase and the physical property tensor which characterize the domains. The classification of tensors and tensor pairs into classes of crystallographically equivalent tensors and tensor pairs, which provides for a classification of domains and domain pairs, is given in Section III. Section IV gives a relationship between the tensors characterizing the domains and the corresponding set of distinct tensors. The central theorem giving the classification off domain pairs for a partial physical property tensor via a classification of ordered distinct tensor pairs is presented in Section V. This theorem determines the number of classes and a representative ordered distinct tensor pair of each class. The ordered distinct tensor pairs belonging to each class are then determined.

II. DOMAINS AND PHYSICAL PROPERTY TENSORS

Let G denote the point group of the high symmetry phase and H the point group of one of the domains. We denote by $D^{(i)}$, $i = 1,2,\ldots,q$ the q domains and by $H^{(i)}$, $i = 1,2,\ldots,q$ the corresponding point groups of each domain. Let **T** denote a spontaneous property tensor which arises at the phase transition. We denote by $T^{(i)}$, $i = 1,2,\ldots,q$ the corresponding specific forms of the tensor **T** characterizing each of the q domains.

The number q of domains is equal to the number of left cosets in the coset decomposition of G with respect to H

$$G = H + g_2H + \ldots + g_qH \tag{2.1}$$

where $g_1 = 1, g_2, \ldots, g_q$ are the coset representatives of this coset decomposition. The number q of domains can also be defined as

$$q = N(G)/N(H) \tag{2.2}$$

where $N(G)$ and $N(H)$ denote the order of the groups G and H, respectively.

We choose domain $D^{(1)}$ as a domain whose point group $H^{(1)}$ is equal to H. The remaining domains are defined by relating their respective point groups $H^{(i)}$, $i = 2,3,\ldots,q$, to the point group $H^{(1)} = H$ via the coset representatives of the coset decomposition (2.1). The domains $D^{(i)}$ and the point groups $H^{(i)}$, $i = 2,3,\ldots,q$, are defined by:

$$D^{(i)} = g_iD^{(1)} \tag{2.3}$$

$$H^{(i)} = g_iHg_i^{-1} \tag{2.4}$$

where g_i, $i = 2,3,\ldots,q$, are the coset representatives of the coset decomposition (2.1). It follows that the specific forms $T^{(i)}$, $i = 1,2,\ldots,q$, of the tensor **T** characterizing each of the domains is related by the same coset representatives to $T^{(1)}$, the form of the tensor **T** characterizing domain $D^{(1)}$. We denote the specific form $T^{(1)}$ of the tensor **T** characterizing domain $D^{(2)}$ by T, and we have

$$T^{(i)} = g_i T \qquad i = 2,3,\ldots,q \tag{2.5}$$

where g_i, $i = 2,3,\ldots,q$ are the coset representatives of the coset decomposition (2.1).

Let G_T denote the stabilizer of T in G. This subgroup G_T of G is the set of all elements g of G which leave T invariant:

$$gT = T \tag{2.6}$$

Since T characterizes domain $D^{(1)}$ whose symmetry group is H, it follows that $hT = T$ for all elements h of H and:

$$H \subseteq G_T \tag{2.7}$$

The stabilizer G_T of the tensor T in G is identical with H or contains H as a proper subgroup. The dichotomy of equation (2.7) distinguishes between tensorial properties which represent full and partial physical property tensors. If $H = G_T$ then the physical property tensor **T** is a full physical property tensor, if $H \subset G_T$ then the physical property tensor **T** is a partial physical property tensor.

For full physical property tensors, where $G_T = H$, the tensors $T^{(i)}$, $i = 1,2,\ldots,q$, are distinct, and we have q distinct domains, i.e., each domain is characterized by a unique form of the tensor T. For partial physical property tensors, where $G_T \supset H$, the tensors $T^{(i)}$, $i = 1,2,\ldots,q$, are not all distinct, and we have $q_T < q$ distinct domains. More than one domain is characterized by a single specific form of the physical property tensor **T**. The number q_T of distinct domains is given by

$$q_T = N(G)/N(G_T) \tag{2.8}$$

and the number d_T of domains with the same form of the physical property tensor **T** is given by:

$$d_T = N(G_T)/N(H) \tag{2.9}$$

From Equations (2.2), (2.8) and (2.9) we have that[14]:

$$q = q_T d_T \tag{2.10}$$

III. CLASSES OF TENSORS AND TENSOR PAIRS

For a given group G and physical property tensor **T**, a set of tensors $T^{(i)}$, $i = 1,2,\ldots,n$, can be partitioned into classes of crystallographically equivalent tensors with respect to G: Two tensors $T^{(i)}$ and $T^{(j)}$ are said to be crystallographically equivalent with respect to G and belong to the same class of crystallographically equivalent tensors with respect to G, if there is an element g of G such that:

$$gT^{(i)} = T^{(j)} \tag{3.1}$$

The set of tensors which constitute a single class of crystallographically equivalent tensors with respect to G is denoted by $G(T^{(i)})$ where $T^{(i)}$ is a representative tensor of that class. All tensors of the class $G(T^{(i)})$ can be generated by applying all elements g of G to the representative tensor $T^{(i)}$. For a given group G, subgroup H, and physical property tensor **T**, the tensors $T^{(i)}$, $i = 1,2, \ldots ,q$, defined by Equation (2.5), constitute a single class of crystallographically equivalent tensors with respect to G denoted by $G(T)$.

For a given group G, subgroup H, and physical property tensor **T** one can define the q^2 ordered tensor pairs $(T^{(i)}), T^{(j)})$, $i,j = 1,2, \ldots ,q$. This set of ordered tensor pairs can be partitioned into classes of crystallographically equivalent ordered tensor pairs with respect to G: Two ordered tensor pairs $(T^{(i)}, T^{(j)})$ and $(T^{(i')}, T^{(j')})$ are said to be crystallographically equivalent with respect to G and belong to the same class of crystallographically equivalent ordered tensor pairs with respect to G, if there is an element g of G such that

$$(T^{(i)}, T^{(j)}) = (gT^{(i')}, gT^{(j')}) \tag{3.2}$$

that is, if $T^{(i)} = gT^{(i')}$ and $T^{(j)} = gT^{(j')}$. The set of ordered tensor pairs which constitute a single class of crystallographically equivalent ordered tensor pairs with respect to G is denoted by $G(T^{(i)}, T^{(j)})$ where $(T^{(i)}, T^{(j)})$ is a representative ordered tensor pair of that class. All ordered tensor pairs belonging to the class $G(T^{(i)}, T^{(j)})$ can be generated by applying all elements g of G to the representative ordered tensor pair $(T^{(i)}, T^{(j)})$.

The tensor and tensor pair classification is also a classification of domains and ordered domain pairs: Two domains $D^{(i)}$ and $D^{(j)}$ belong to the same class of crystallographically equivalent domains with respect to G and **T** if the corresponding tensors $T^{(i)}$ and $T^{(j)}$ are crystallographically equivalent tensors with respect to G. Two ordered domain pairs $(D^{(i)}, D^{(j)})$ and $(D^{(i')}, D^{(j')})$ belong to the same class of crystallographically equivalent ordered domain pairs with respect to G and **T** if the corresponding ordered tensor pairs $(T^{(i)}, T^{(j)})$ and $(T^{(i')}, T^{(j')})$ are crystallographically equivalent ordered tensor pairs with respect to G.

IV. DISTINCT TENSORS

Let $T_d^{(a)}$, $a = 1,2, \ldots ,q_T$, denote the $q_T \leq q$ distinct forms of the tensor **T** among the q tensors $T^{(i)}$, $i = 1,2, \ldots ,q$, characterizing the q domains $D^{(i)}$, $i = 1,2, \ldots ,q$. We choose $T_d^{(1)} = T^{(1)} = T$ and define $T_d^{(a)}$, $a = 2,3, \ldots ,q_T$, by

$$T_d^{(a)} = \bar{g}_a T \qquad a = 2,3, \ldots ,q_T \tag{4.1}$$

where the $\bar{g}_a$ are the coset representatives in the coset decomposition of G with respect to G_T:

$$G = G_T + \bar{g}_2 G_T + \ldots \bar{g}_{q_T} G_T \tag{4.2}$$

The tensor $T^{(i)}$ characterizing domain $D^{(i)}$ is the distinct tensor $T_d^{(a)}$ if the i^{th} coset $g_i H$ of the coset decomposition (2.1) of G with respect to H is contained in the a^{th} coset $\bar{g}_a G_T$ of the coset decomposition (4.2) of G with respect to G_T: From equation (2.4)

$$T^{(i)} = g_i h T = g_i T$$

and

$$T^{(i)} = \bar{g}_a g_T T$$

where g_T is an element of G_T and g_i is taken as an element of the a^{th} coset of the coset decomposition (4.2). Consequently:

$$T^{(i)} = \bar{g}_a T = T_d^{(1)}$$

For example, we consider the case where G = m3m, H is the subgroup 4_x, and a polar vector physical property tensor. In Table I we list the elements of each coset in the coset decomposition, Equation (2.1), of G with respect to H. There are $q = 12$ cosets and the coset representatives g_i, $i = 1,2, \ldots ,12$, are chosen as the first element in each row of coset elements in Table I. In Table II, we index each of the 12 cosets, list the coset representatives g_i, the point groups $H^{(i)}$ defined by Equation (2.4), and the polar vector tensors $T^{(i)}$ defined by Equation (2.5).

In Table III, we list the elements of each coset in the coset decomposition, Equation (4.2), of G with respect to the stabilizer $G_T = 4_x m_y m_{yz}$ of T in G. There are $q_T = 6$ cosets and the coset representatives $\bar{g}_a$, $a = 1,2, \ldots ,6$, are chosen as the first element in each row of coset elements in Table III. In Table IV, we index each of the 6 cosets, list the coset representatives $\bar{g}_a$, and the distinct tensors $T_d^{(a)}$ of Equation (4.1). The correlation between the distinct tensors $T_d^{(a)}$ and the individual domains' characteristic tensors $T^{(i)}$ is also given: The indexes i of tensors

TABLE I

Coset decomposition of G = m3m with respect to $H = 4x$. Each row lists the elements of a left coset $g_i H$, $i = 1,2, \ldots ,12$. We choose the coset representatives g_i as the first element in each coset.

1	2_x	4_x	4_x^3
$\bar{1}$	m_x	$\bar{4}_x$	$\bar{4}_x^3$
2_z	2_y	2_{yz}	$2_{\bar{y}z}$
m_z	m_y	m_{yz}	$m_{\bar{y}z}$
2_{xz}	4_y^3	3_{xyz}^2	$3_{x\bar{y}z}$
$2_{\bar{x}z}$	4_y	$3_{\bar{x}yz}$	$3_{xy\bar{z}}$
$2_{\bar{x}y}$	4_z^3	$3_{x\bar{y}z}^2$	$3_{\bar{x}yz}^2$
2_{xy}	4_z	$3_{xy\bar{z}}^2$	3_{xyz}
m_{xz}	$\bar{4}_y^3$	$\bar{3}_{xyz}^5$	$\bar{3}_{x\bar{y}z}$
$m_{\bar{x}z}$	$\bar{4}_y$	$\bar{3}_{\bar{x}yz}$	$\bar{3}_{xy\bar{z}}$
$m_{\bar{x}y}$	$\bar{4}_z^3$	$\bar{3}_{x\bar{y}z}^5$	$\bar{3}_{\bar{x}yz}^5$
m_{xy}	$\bar{4}_z$	$\bar{3}_{xy\bar{z}}^5$	$\bar{3}_{xyz}$

TABLE II

Listed are the indexes i of the cosets of the coset decomposition of G = m3m with respect to H = $4x$, coset representatives g_i, the i^{th} domain's symmetry group $H^{(i)}$, and the specific form $T^{(i)}$ of the polar vector tensorial property tensor T.

i	g_i	$H^{(i)}$	$T^{(i)}$
1	1	4_x	(A,0,0)
2	$\bar{1}$	4_x	(−A,0,0)
3	2_z	4_x	(−A,0,0)
4	m_z	4_x	(A,0,0)
5	2_{xz}	4_z	(0,0,A)
6	$2_{\bar{x}z}$	4_z	(0,0,−A)
7	$2_{\bar{x}y}$	4_y	(0,−A,0)
8	2_{xy}	4_y	(0,A,0)
9	m_{xz}	4_z	(0,0,−A)
10	$m_{\bar{x}z}$	4_z	(0,0,A)
11	$m_{\bar{x}y}$	4_y	(0,A,0)
12	m_{xy}	4_y	(0,−A,0)

$T^{(i)}$ identical with the distinct tensor $T_d^{(a)}$ are given in the last column of row "a" of Table IV. These have been determined by finding all cosets g_iH of the coset decomposition of G with respect to H, Equation (2.1) and Table I, contained in the coset $\bar{g}_aG_T$ of the coset decomposition of G with respect to G_T, Equation (4.2) and Table III. For example, for $a = 1$, $i = 1,4$, since both cosets g_1H and g_4H, see Table I, are contained in the coset $\bar{g}_1G_T$, see Table III.

V. DISTINCT TENSOR PAIRS

To each domain there corresponds a distinct tensor $T_d^{(a)}$ and to each ordered domain pair there corresponds an ordered distinct tensor pair $(T_d^{(a)},T_d^{(b)})$. All

TABLE III

Coset decomposition of G = m3m with respect to the stabilizer $G_T = 4_xm_ym_{yz}$ of the polar vector **T** in G. Each row lists the elements of the left cosets $\bar{g}_aG_T$, = 1,2, . . . , 6. We choose the coset representatives $\bar{g}_a$ as the first element of each coset. Dashed lines separate sets of cosets which constitute double cosets of the double coset decomposition of G with respect to G_T. We choose the double coset representatives $g_k^{(dc)}$ as the first element of each double coset.

1	2_x	4_x	4_x^3	m_z	m_y	m_{yz}	$m_{\bar{y}z}$
$\bar{1}$	m_x	$\bar{4}_x$	$\bar{4}_x^3$	2_z	2_y	2_{yz}	$2_{\bar{y}z}$
2_{xz}	4_y^3	3_{xyz}^2	$3_{x\bar{y}z}$	$\bar{4}_y$	$m_{\bar{x}z}$	$\bar{3}_{xy\bar{z}}$	$\bar{3}_{\bar{x}yz}$
$2_{\bar{x}z}$	4_y	$3_{\bar{x}yz}$	$3_{xy\bar{z}}$	$\bar{4}_y^3$	m_{xz}	$\bar{3}_{x\bar{y}z}$	$\bar{3}_{xyz}^5$
$2_{\bar{x}y}$	4_z^3	$3_{x\bar{y}z}^2$	$3_{\bar{x}yz}^2$	m_{xy}	$\bar{4}_z$	$\bar{3}_{xy\bar{z}}^5$	$\bar{3}_{xyz}$
2_{xy}	4_z	$3_{xy\bar{z}}^2$	3_{xyz}	$m_{\bar{x}y}$	$\bar{4}_z^3$	$\bar{3}_{x\bar{y}z}^5$	$\bar{3}_{\bar{x}yz}^5$

TABLE IV

Listed are the indexes "a" of the cosets of the coset decomposition of G = m3m with respect to $G_T = 4_x m_y m_{yz}$, the coset representatives $\bar{g}_a$, the corresponding distinct tensor $T_d^{(a)}$, and the indexes of the cosets $g_i H$ of the coset decomposition of G with respect to $H = 4_x$ contained in each coset $\bar{g}_a G_T$ of the coset decomposition of G with respect to G_T.

a	$\bar{g}_a$	$T_d^{(a)}$	i
1	1	(A,0,0)	1,4
2	$\bar{1}$	(−A,0,0)	2,3
3	2_{xz}	(0,0,A)	5,10
4	$2_{\bar{x}z}$	(0,0,−A)	6,9
5	$2_{\bar{x}y}$	(0,−A,0)	7,12
6	2_{xy}	(0,A,0)	8,11

ordered distinct tensor pairs can be partitioned into classes of crystallographically equivalent ordered distinct tensor pairs as are all ordered tensor pairs in Section III. Two ordered distinct tensor pairs $(T_d^{(a)}, T_d^{(b)})$ and $(T_d^{(a')}, T_d^{(b')})$ are said to be crystallographically equivalent with respect to G, and to belong to the same class of ordered distinct tensor pairs, if there is an element g of G such that

$$(T_d^{(a)}, T_d^{(b)}) = (gT_d^{(a')}, gT_d^{(b')}) \tag{5.1}$$

that is, if $T_d^{(a)} = gT_d^{(a')}$ and $T_d^{(b)} = gT_d^{(b')}$.

The number of classes of ordered distinct tensor pairs $(T_d^{(a)}, T_d^{(b)})$ is the same as the number of classes of tensor pairs $(T^{(i)}, T^{(j)})$. This number of classes is determined by the following theorem:

Let G be the point group of the high symmetry phase, H the point group of a domain, and T *the specific form of the physical tensor* **T** invariant under *H*. *The number* N *of crystallographically equivalent ordered distinct tensor pair classes is equal to the number of double cosets in the double coset decomposition of G with respect to* G_T

$$G = G_T e G_T + G_T g_2^{(dc)} G_T + \ldots + G_T g_N^{(dc)} G_T \tag{5.2}$$

where G_T *is the stabilizer of* T *in G and* $g_k^{(dc)}$, k = 1,2, . . . ,N, *are the double coset representatives. A representative ordered distinct tensor pair of each class of crystallographically equivalent ordered distinct tensor pairs is given by* $(T, g_k^{(dc)}T)$, k = 1,2,. . . ,N.

Proof: In each class $G(T_d^{(a)}, T_d^{(b)})$ there is at least one ordered distinct tensor pair of the form $(T, T_d^{(c)})$: From Equation (4.1) $T_d^{(a)} = \bar{g}_a T$, and $(\bar{g}_a^{-1}T_d^{(a)}, \bar{g}_a^{-1}T_d^{(b)})$ is also an ordered distinct tensor pair in the class $G(T^{(a)}, T_d^{(b)})$. It follows that

$$\begin{aligned}(\bar{g}_a^{-1}T_d^{(a)}, \bar{g}_a^{-1}T_d^{(b)}) &= (\bar{g}_a^{-1}\bar{g}_a T, \bar{g}_a^{-1}\bar{g}_b T)\\ &= (T, \bar{g}_c g_T T)\\ &= (T, \bar{g}_c T)\\ &= (T, T_d^{(c)})\end{aligned}$$

where $\bar{g}_a^{-1}\,\bar{g}_b = \bar{g}_c g_T$ and g_T is an element of G_T. Since the number of double cosets in Equation (5.2) is less than or equal to the number of cosets in Equation (4.2), the number of classes is then $N \leq q_T$, less than or equal to the number of distinct tensors $T_d^{(c)}$. The number N of classes is equal to the number of classes of ordered distinct tensor pairs of the form $(T, T_d^{(c)})$.

Two ordered distinct tensor pairs $(T, T_d^{(a)})$ and $(T, T_d^{(a')})$ belong to the same class of ordered distinct tensor pairs if and only if there is an element g of G such that $gT = T$ and $gT_d^{(a')} = T_d^{(a)}$, that is, if and only if g is an element of G_T and $T_d^{(a)}$ and $T_d^{(a')}$ belong to the same class of ordered distinct tensors $G_T(T_d^{(a)})$. Consequently, the number N of classes of ordered distinct tensor pairs equals the number of classes $G_T(T_d^{(a)})$ in a partition of all distinct tensors $T_d^{(a)}$ into classes with respect to G_T. The class $G(T)$ contains all distinct tensors $T_d^{(a)}$. Consequently $G(T)$ can be partitioned into a sum of the N classes $G_T(T_d^{(a)})$:

$$G(T) = G_T(T) + G_T(T_d^{(2)}) + \ldots + G_T(T_d^{(N)}) \tag{5.3}$$

The elements of G which when applied to T give rise to distinct tensors of the class $G_T(T_d^{(a)})$ are found by noting that:

$$G_T(T_d^{(a)}) = G_T(\bar{g}_a T) = G_T\bar{g}_a(T) = G_T\bar{g}_a G_T(T) \tag{5.4}$$

That is, the set of elements of the double coset $G_t\bar{g}_a G_T$ generate from T all distinct tensors of the class $T_T(T_d^{(a)})$. Consequently, there is a one-to-one correspondence between the double cosets of the double coset decomposition of G with respect to G_T and the classes $G_T(T_d^{(a)})$. The number N of crystallographically equivalent ordered distinct tensor pairs is then equal to the number of double cosets in the double coset decomposition of G with respect to G_T given in Equation (5.2). It follows from Equation (5.4) that since $g_k^{(dc)}$ is an element of the double coset $G_T g_k^{(dc)} G_T$, that a representative ordered distinct tensor pair of each class is given by $(T, g_k^{(dc)}T)$ for $k = 1, 2, \ldots, N$. *q.e.d.*

The above theorem is independent of which domain's point group is chosen as the subgroup H of G. The number N of classes of crystallographically equivalent ordered distinct tensor pairs remains the same, however the theorem's representative ordered distinct tensor pairs may change: Let $H' \neq H$ be a point group of a domain. By Equations (2.4) and (2.5) we have that $H' = g_i H g_i^{-1}$ and $T' = g_i T$ for some specific value of the index i. It follows that G_T', the stabilizer of T' in G, is $G_T' = g_i G_T g_i^{-1}$. By rewriting Equation (5.2) as $G = g_i G g_i^{-1}$ one obtains that the double coset decomposition of G with respect to G_T' contains N double cosets with double coset representatives $g_k^{(dc)'} = g_i g_k^{(dc)} g_i^{-1}$. The representative ordered distinct tensor pairs

$$\begin{aligned}(T', g_k^{(dc)'}T') &= (g_i T, g_i g_k^{(dc)} g_i^{-1} g_i T) \\ &= g_i(T, g_k^{(dc)}\mathrm{T})\end{aligned}$$

that is, for each k, $(T', g_k^{(dc)'}T')$ and $(T, g_k^{(dc)}T)$ are ordered distinct tensor pairs belonging to the same class of crystallographically equivalent ordered distinct tensor pairs. Consequently, replacing the subgroup H of G in the above theorem by the conjugate subgroup H' of G leads to a change in the theorem's representative ordered distinct tensor pair of each class.

As an example, we take the case where G = m3m, $H = 4_x$, and a polar vector property tensor **T**, considered in Section IV. The stabilizer of T in G is $G_T = 4_x m_y m_{yz}$. The coset decomposition of G with respect to G_T, Equation (4.2), is given in Table III. The double coset decomposition of G with respect to G_T, Equation (5.2), is also given in Table III[15]: Dashed horizontal lines separate sets of cosets of Equation (4.2) which are contained in the same double coset of Equation (5.2). From Table III, we have that there are 3 double cosets in the double coset decomposition of G with respect to G_T. We choose the first element of each double coset in Table III as the double coset representative $g_k^{(dc)}$. In Table V we index the double cosets, give the double coset representatives $g_k^{(dc)}$, and give the representative ordered distinct tensor pair $(T, g_k^{(dc)}T)$ for each of the $N = 3$ classes of crystallographically equivalent ordered distinct tensor pairs.[16]

Since there are $q_T = 6$ distinct tensors in this case, see Table IV, there are $q_T^2 = 36$ distinct ordered tensor pairs which are classified into three classes of crystallographically equivalent distinct ordered tensor pairs. A distinct ordered tensor pair $(T_d^{(a)}, T_d^{(b)})$ belongs to the class whose representative distinct ordered tensor pair is $(T, g_k^{(dc)}T)$ if the product of the coset representatives $\bar{g}_a^{-1}\bar{g}_b$ is an element of the k^{th} double coset of the double coset decomposition of G with respect to G_T:

$$(T_d^{(a)}, T_d^{(b)}) = (\bar{g}_a T, \bar{g}_b T)$$
$$= \bar{g}_a(T, \bar{g}_a^{-1}\bar{g}_b T)$$

Writing $\bar{g}_a^{-1}\bar{g}_b$ as the element $g_T g_k^{(dc)} g_T'$ of the double coset decomposition, we have

$$= \bar{g}_a(T, g_T g_k^{(dc)} g_T' T)$$
$$= \bar{g}_a g_T(g_T^{-1}T, g_k^{(dc)} g_T' T)$$

and since T is invariant under elements of G_T

$$(T_d^{(a)}, T_d^{(b)}) = \bar{g}_a g_T(T, g_k^{(dc)}T)$$

and $(T_d^{(a)}, T_d^{(b)})$ belongs to the class whose representative ordered tensor pair is $(T, g_k^{(dc)}T)$. In Table VI we have tabulated the ordered distinct tensor pairs $(T_d^{(a)}, T_d^{(b)})$ belonging to each class of crystallographically equivalent ordered distinct tensor

TABLE V

Listed are the indexes $k = 1,2,3$ of the double coset decomposition of G = m3m with respect to $G_T = 4_x m_y m_{yz}$, the double coset representatives $g_k^{(dc)}$, and the representative ordered distinct tensor pair $(T, g_k^{(dc)}T)$ of each class of crystallographically equivalent ordered distinct polar vector tensor pairs.

k	$g_k^{(dc)}$	$(T, g_k^{(dc)}T)$
1	1	((A, 0, 0),(A, 0, 0))
2	2_z	((A, 0, 0),(−A, 0, 0))
3	2_{xz}	((A, 0, 0),(0, 0, A))

TABLE VI

Listed are the indexes $k = 1,2,3$ of the classes of crystallographically equivalent ordered distinct polar vector tensor pairs along with the pairs of indexes (a,b) of each ordered distinct tensor pair $(T_d^{(a)},T_d^{(b)})$ belonging to each class.

k	(a,b)
1	(1,1), (2,2), (3,3), (4,4), (5,5), (6,6).
2	(1,2), (2,1), (3,4), (4,3), (5,6), (6,5).
3	(1,3), (1,4), (1,5), (1,6), (2,3), (2,4).
	(2,5), (2,6), (3,1), (3,2), (3,5), (3,6).
	(4,1), (4,2), (4,5), (4,6), (5,1), (5,2).
	(5,3), (5,4), (6,1), (6,2), (6,3), (6,4).

pairs, by listing the pairs of indexes (a,b) alongside the corresponding index k of the representative ordered distinct tensor pair $(T,g_k^{(dc)}T)$. We note that for this particular case of a polar vector, the three classes of crystallographically equivalent ordered distinct tensor pairs can be interpreted as the three classes of parallel, anti-parallel, and perpendicular polar vector pairs.

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15. Tables of the elements of the left cosets and double cosets in the left coset and double coset decompositions of the 32 crystallographic point groups G with respect to one of each set of conjugate subgroups H in G, see for example Table III for the case G = m3m and $H = 4_x m_y m_{yz}$, has been given by V. Janovec and E. Dvorakova (Report V-FZU 75/1, Institute of Physics, Czechoslovak Academy of Sciences, Prague, Czechoslovakia, 1974). These tables have been extended and re-tabulated in International notation to include all subgroups of the thirty-two crystallographic point groups (T. Wike, D. B. Litvin, V. Janovec and E. Dvorakova, unpublished, 1988).
16. Tables of the representative distinct ordered tensor pairs $(T,g_k^{(dc)}T)$ of each class of crystallographically equivalent distinct ordered tensor pairs for polar and axial vectors **T** for all crystallographic point groups G and subgroups H of G have been tabulated. (T. Wike and D. B. Litvin, unpublished, 1988).