THE EVOLUTION OF...

Edited by **Abe Shenitzer and John Stillwell**

Mathematics in the 20th Century[1]

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If you talk about the end of one century and the beginning of the next you have two choices, both of them difficult. One is to survey the mathematics over the past hundred years; the other is to predict the mathematics of the next hundred years. I have chosen the more difficult task. Everybody can predict and we will not be around to find out whether we were wrong. But giving an impression of the past is something that everybody can disagree with.

All I can do is give you a personal view. It is impossible to cover everything, and in particular I will leave out significant parts of the story, partly because I am not an expert, and partly because they are covered elsewhere. I will say nothing, for example, about the great events in the area between logic and computing associated with the names of people like Hilbert, Gödel, and Turing. Nor will I say much about the applications of mathematics, except in fundamental physics, because they are so numerous and they need such special treatment. Each would require a lecture to itself. Moreover, there is no point in trying to give just a list of theorems or even a list of famous mathematicians over the last hundred years. That would be rather a dull exercise. So instead I am going to try and pick out some themes that I think run across the board in many ways and underline what has happened.

Let me first make a general remark. Centuries are crude numbers. We do not really believe that after a hundred years something suddenly stops and starts again. So when I describe the mathematics of the 20th century, I am going to be rather cavalier about dates. If something started in the 1890s and moved into the 1900s, I shall ignore such detail. I will behave like an astronomer and work in rather approximate numbers. In fact, many things started in the 19th century and only came to fruition in the 20th century.

One of the difficulties of this exercise is that it is very hard to put oneself back in the position of what it was like in 1900 to be a mathematician, because so much of the mathematics of the last century has been absorbed by our culture, by us. It is very hard to imagine a time when people did not think in our terms. In fact, if you make a really important discovery in mathematics you will get omitted altogether! You simply get absorbed into the background. So going back, you have to try to imagine what it was like in a different era when people did not think in our way.

1. LOCAL TO GLOBAL. I am going to start by listing some themes and talking around them. My first theme is broadly under what you might call the passage from the local to the global. In the classical period people on the whole would have studied things on a small scale, in local coordinates and so on. In this century, the emphasis has shifted to try and understand the global, large-scale behavior. And because global behavior is more difficult to understand, much of it is done qualitatively, and topological ideas become very important. It was Poincaré who both made the pioneering

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steps in topology and forecast that topology would be an important ingredient in 20th-century mathematics. Incidentally, Hilbert, who made his famous list of problems, did not. Topology hardly figured in his list of problems. But for Poincaré it was quite clear that it would be an important factor.

Let me try to list a few of the areas and you can see what I have in mind. Consider, for example, complex analysis ("function theory", as it was called), which was at the center of mathematics in the 19th century, the work of great figures like Weierstrass. For them, a function was a function of one complex variable and for Weierstrass a function was a power series, something you could lay your hands on, write down, and describe explicitly; or a formula. Functions were formulas: they were explicit things. But then the work of Abel, Riemann, and subsequent people moved us away, so that functions became defined not just by explicit formulas but more by their global properties: by where their singularities were, where their domains of definition were, where they took their values. These global properties were the distinguishing characteristic feature of the function. The local expansion was only one way of looking at it.

A similar sort of story occurs with differential equations. Originally, to solve a differential equation people would have looked for an explicit local solution: something you could write down and lay your hands on. As things evolved, solutions became implicit. You could not necessarily describe them in nice formulas. The singularities of the solution were the things that really determined its global properties. This is very much similar in spirit, but different in detail, to what happened in complex analysis.

In differential geometry, the classical work of Gauss and others would have described small pieces of space, small bits of curvature and the local equations that describe local geometry. The shift from there to the large scale is a rather natural one, where you want to understand the global overall picture of curved surfaces and the topology that goes with them. When you move from the small to the large, the topological features become the ones that are most significant.

Although it does not apparently fit into the same framework, number theory shared a similar development. Number theorists distinguish what they call the "local theory", where they talk about a single prime, one prime at a time, or a finite set of primes, and the "global theory", where you consider all primes simultaneously. This analogy between primes and points, between the local and global, has had an important effect in the development of number theory, and the ideas that have taken place in topology have had their impact on number theory.

In physics, of course, classical physics is concerned with the local story, where you write down the differential equation that governs the small-scale behavior; and then you have to study the large-scale behavior of a physical system. All physics is concerned really with predicting what will happen when you go from a small scale, where you understand what is happening, to a large scale, and follow through to the conclusions.

2. INCREASE IN DIMENSIONS. My second theme is different. It is what I call the increase in dimensions. Again, we start with the classical theory of complex variables: classical complex variable theory was primarily the theory of one complex variable studied in detail, with great refinement. The shift to two or more variables fundamentally took place in this century, and in that area new phenomena appear. Not everything is just the same as in one variable. There are quite new features, and the theory of n variables has become more and more dominant, one of the major success stories of this century.

Again, differential geometers in the past would have studied primarily curves and surfaces. We now study the geometry of n-dimensional manifolds, and you have to

think carefully to realize that this was a major shift. In the early days, curves and surfaces were things you could really see in space. Higher dimensions were slightly fictitious, things that you could imagine mathematically, but perhaps you did not take them seriously. The idea that you took these things seriously and studied them to an equal degree is really a product of the 20th century. Also, it would not have been nearly so obvious to our 19th-century predecessors to think of increasing the number of functions, to study not only one function but several functions, or vector-valued functions. So we have seen an increase in the number both of independent and dependent variables.

Linear algebra was always concerned with more variables, but there the increase in dimension was to be more drastic. It went from finite dimensions to infinite dimensions, from linear space to Hilbert space, with an infinite number of variables. There was, of course, analysis involved. After functions of many variables, you can have functions of functions, functionals. These are functions on the space of functions. They all have essentially infinitely many variables, and that is what we call the calculus of variations. A similar story was developing with general (non-linear) functions, an old subject, but one that really was coming into prominence in the 20th century. So that is my second theme.

3. COMMUTATIVE TO NON-COMMUTATIVE.
A third theme is the shift from commutative to non-commutative. This is perhaps one of the most characteristic features of mathematics, particularly algebra, in the 20th century. The non-commutative aspect of algebra has been extremely prominent, and, of course, its roots are in the 19th century. It has diverse roots. Hamilton's work on quaternions was probably the single biggest surprise and had a major impact, motivated in fact by ideas having to do with physics. There was the work of Grassmann on exterior algebras—another algebraic system that has now been absorbed in our theory of differential forms. Of course, the work of Cayley on matrices, based on linear algebra, and that of Galois, based on group theory, were other highlights.

All these are different ways or strands that form the basis of the introduction of non-commutative multiplication into algebra, which is the bread and butter of 20th-century algebraic machinery. We do not think anything of it, but in the 19th century all these foregoing examples were, in their different ways, tremendous breakthroughs. Of course, the applications of these ideas came quite surprisingly in different directions. The applications of matrices and non-commutative multiplication in physics came with quantum theory. The Heisenberg commutation relations are a most important example of a significant application of non-commutative algebra in physics, subsequently extended by von Neumann into his theory of algebras of operators.

Group theory has also been a dominant feature of the 20th century and I shall return to this later.

4. LINEAR TO NON-LINEAR.
My next theme is the passage from the linear to the non-linear. Large parts of classical mathematics are either fundamentally linear or, if not exactly linear, approximately linear, studied by some sort of perturbation expansion. The really non-linear phenomena are much harder, and have only been seriously tackled in this century.

The story starts off with geometry: Euclidean geometry, geometry of the plane, of space, of straight lines, everything linear; and then through various stages of non-Euclidean geometry to Riemann's more general geometry, where things are fundamentally non-linear. In differential equations, the serious study of non-linear phenomena has thrown up a whole range of new phenomena that you do not see in the classical

treatments. I might just pick out two here, solitons and chaos, two very different aspects of the theory of differential equations that have become extremely prominent and popular in this century. They represent alternative extremes. Solitons represent unexpectedly organized behavior of non-linear differential equations, and chaos represents unexpectedly disorganized behavior. Both of them are present in different regimes, and are interesting and important, but they are fundamentally non-linear phenomena. Again, you can trace back the early history of some of the work on solitons into the last part of the 19th century, but only very slightly.

In physics, of course, Maxwell's equations, the fundamental equations of electromagnetism, are linear partial differential equations. Their counterparts, the famous Yang-Mills equations, are non-linear equations that are supposed to govern the forces involved in the structure of matter. The equations are non-linear, because the Yang-Mills equations are essentially matrix versions of Maxwell's equations, and the fact that matrices do not commute is what produces the non-linear term in the equations. So here we see an interesting link between non-linearity and non-commutativity. Non-commutativity does produce non-linearity of a particular kind, and this is particularly interesting and important.

5. GEOMETRY VERSUS ALGEBRA.
So far I have picked out a few general themes. I want now to talk about a dichotomy in mathematics that has been with us all the time, oscillating backwards and forwards, and gives me a chance to make some philosophical speculations or remarks. I refer to the dichotomy between geometry and algebra. Geometry and algebra are the two formal pillars of mathematics, and both are very ancient. Geometry goes back to the Greeks and before; algebra goes back to the Arabs and the Indians, so they have both been fundamental to mathematics, but they have had an uneasy relationship.

Let me start with the history of the subject. Euclidean geometry is the prime example of a mathematical theory, and it was firmly geometrical until the introduction by Descartes of algebraic coordinates in what we now call the Cartesian plane. That was an attempt to reduce geometrical thinking to algebraic manipulation. This was, of course, a big breakthrough or a big attack on geometry from the side of the algebraists. If you compare in analysis the work of Newton and Leibniz, they belong to different traditions: Newton was fundamentally a geometer, Leibniz was fundamentally an algebraist, and there were good, profound reasons for that. For Newton, geometry, or the calculus as he developed it, was the mathematical attempt to describe the laws of nature. He was concerned with physics in a broad sense, and physics took place in the world of geometry. If you wanted to understand how things worked, you thought in terms of the physical world, you thought in terms of geometrical pictures. When he developed the calculus, he wanted to develop a form of it that would be as close as possible to the physical context behind it. He therefore used geometrical arguments, because that was keeping close to the meaning. Leibniz, on the other hand, had the aim, the ambitious aim, of formalizing the whole of mathematics, turning it into a big algebraic machine. This was totally opposed to the Newtonian approach. They also used very different notations. As we know, in the big controversy between Newton and Leibniz, Leibniz's notation won out. We have followed his way of writing derivatives. Newton's spirit is still there, but it got buried for a long time.

By the end of the 19th century, a hundred years ago, the two major figures were Poincaré and Hilbert. I have mentioned them already, and they are, very crudely speaking, disciples of Newton and Leibniz respectively. Poincaré's thought was more in the spirit of geometry, topology, using those ideas as a fundamental insight. Hilbert was more a formalist; he wanted to axiomatize, formalize, and give rigorous, formal, pre-

sentations. They clearly belong to different traditions, though any great mathematician cannot be easily categorized.

When preparing this talk, I thought I should put down some further names from our present generation who represent the continuation of these traditions. It is very difficult to talk about living people—whom to put on the list? I then thought to myself: who would mind being put on either side of such a famous list? I have, therefore, chosen two names: Arnol´d as the inheritor of the Poincaré-Newton tradition, and Bourbaki as, I think, the most famous disciple of David Hilbert. Arnol´d makes no bones about the fact that his view of mechanics, in fact, of physics, is that it is fundamentally geometrical, going back to Newton; everything in between, with the exception of a few people like Riemann, who was a bit of a digression, was a mistake. Bourbaki tried to carry on the formal program of Hilbert of axiomatizing and formalizing mathematics to a remarkable extent, with some success. Each point of view has its merits, but there is tension between them.

Let me try to explain my own view of the difference between geometry and algebra. Geometry is, of course, about space; of that there is no question. If I look out at the audience in this room I can see a lot, in one single second or microsecond I can take in a vast amount of information and that is, of course, not an accident. Our brains have been constructed in such a way that they are extremely concerned with vision. Vision, I understand from friends who work in neurophysiology, uses up something like 80 or 90 percent of the cortex of the brain. There are about 17 different centers in the brain, each of which is specialized in a different part of the process of vision: some parts are concerned with vertical, some parts with horizontal, some parts with colour, perspective, finally some parts are concerned with meaning and interpretation. Understanding, and making sense of, the world that we see is a very important part of our evolution. Therefore spatial intuition or spatial perception is an enormously powerful tool, and that is why geometry is actually such a powerful part of mathematics—not only for things that are obviously geometrical, but even for things that are not. We try to put them into geometrical form because that enables us to use our intuition. Our intuition is our most powerful tool. That is quite clear if you try to explain a piece of mathematics to a student or a colleague. You have a long, difficult argument and finally the student understands. What does the student say? The student says, "I see!" Seeing is synonymous with understanding, and we use the word "perception" to mean both things as well. At least this is true of the English language. It would be interesting to compare this with other languages. I think it is very fundamental that the human mind has evolved with this enormous capacity to absorb a vast amount of information by instantaneous visual action, and mathematics takes that and perfects it.

Algebra, on the other hand (and you may not have thought about it like this), is concerned essentially with time. Whatever kind of algebra you are doing, a sequence of operations is performed one after the other, and "one after the other" means you have got to have time. In a static universe you cannot imagine algebra, but geometry is essentially static. I can just sit here and see, and nothing may change, but I can still see. Algebra, however, is concerned with time, because you have operations that are performed sequentially and, when I say "algebra", I do not just mean modern algebra. Any algorithm, any process for calculation, is a sequence of steps performed one after the other, the modern computer makes that quite clear. The modern computer takes its information in a stream of zeros and ones and gives the answer.

Algebra is concerned with manipulation in *time*, and geometry is concerned with *space*. These are two orthogonal aspects of the world, and they represent two different points of view in mathematics. Thus the argument or dialogue between mathematicians

in the past about the relative importance of geometry and algebra represents something very fundamental.

Of course, it does not pay to think of this as an argument in which one side loses and the other side wins. I like to think of it in the form of an analogy: "Should you just be an algebraist or a geometer?" is like saying "Would you rather be deaf or blind?" If you are blind, you do not see space, if you are deaf, you do not hear, and hearing takes place in time. On the whole, we prefer to have both faculties.

In physics, there is an analogous, roughly parallel, division between the concepts and the experiments. Physics has two parts to it: *theory*—concepts, ideas, words, laws—and *experimental apparatus*. I think that concepts are in some broad sense geometrical, since they are concerned with things taking place in the real world. An experiment, on the other hand, is more like an algebraic computation. You do something in time; you measure some numbers; you insert them into formulas, but the basic concepts behind the experiments are a part of the geometrical tradition.

One way to put the dichotomy in a more philosophical or literary framework is to say that algebra is to the geometer what you might call the "Faustian Offer". As you know, Faust in Goethe's story was offered whatever he wanted by the devil in return for selling his soul. Algebra is the offer made by the devil to the mathematician. The devil says: "I will give you this powerful machine, and it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvellous machine." [Nowadays you can think of it as a computer!] Of course we like to have things both ways: we would probably cheat on the devil, pretend we are selling our soul, and not give it away. Nevertheless the danger to our soul is there, because when you pass over into algebraic calculation, essentially you stop thinking; you stop thinking geometrically, you stop thinking about the meaning.

I am a bit hard on the algebraists here, but fundamentally the purpose of algebra always was to produce a formula that one could put into a machine, turn a handle and get the answer. You took something that had a meaning; you converted it into a formula; and you got out the answer. In that process you do not need to think any more about what the different stages in the algebra correspond to in the geometry. You lose the insights and this can be important at different stages. You must not give up the insight altogether! You might want to come back to it later on. That is what I mean by the Faustian Offer. I am sure it is provocative.

This choice between geometry and algebra has led to hybrids that confuse the two, and the division between algebra and geometry is not as straightforward and naive as I just said. For example, algebraists frequently will use diagrams. What is a diagram except a concession to geometrical intuition?

6. TECHNIQUES IN COMMON. Let me go back now to talk not so much about themes in terms of content, but perhaps in terms of techniques and common methods that have been used. I want to describe a number of common methods that have been applied in a whole range of fields. The first is

Homology Theory. Homology theory starts off traditionally as a branch of topology. It is concerned with the following situation. You have a complicated topological space and you want to extract from it some simple information that involves counting holes or something similar, some additive linear invariants you can associate to a complicated space. It is a construction, if you like, of linear invariants in a non-linear situation. Geometrically, you think of cycles that you can add and subtract and then you get what is called the homology group of a space. Homology is a fundamental algebraic tool that

was invented in the first half of the century as a way of getting some information about topological spaces; some algebra extracted out of the geometry.

Homology also appears in other contexts. Another source of homology theory goes back to Hilbert and the study of polynomials. Polynomials are functions that are not linear, and you can multiply them to get higher degrees. It was Hilbert's great insight to consider "ideals", linear combinations of polynomials, with common zeros. He looked for generators of these ideals. Those generators might be redundant. He looked at the relations and then for relations between the relations. He got a hierarchy of such relations, which were called "Hilbert syzygies", and this theory of Hilbert was a very sophisticated way of trying to reduce a non-linear situation, the study of polynomials, to a linear situation. Essentially, Hilbert produced a complicated system of linear relations that encapsulates some of the information about non-linear objects, the polynomials.

This algebraic theory is in fact very parallel to the topological theory, and they have now got fused together into what is called "homological algebra". In algebraic geometry, one of the great triumphs of the 1950s was the development of the cohomology theory of sheaves and its extension to analytic geometry by the French school of Leray, Cartan, Serre, and Grothendieck, where you have a combination of the topological ideas of Riemann-Poincaré, the algebraic ideas of Hilbert, and some analysis thrown in for good measure.

It turns out that homology theory has wider applications still, in other branches of algebra. You can introduce homology groups, which are always linear objects associated to non-linear objects. You can take groups, for example finite groups, or Lie algebras: both have homology groups associated to them. In number theory there are very important applications of homology theory, through the Galois group. So homology theory has turned out to be one of the powerful tools to analyze a whole range of situations, a typical characteristic of 20th-century mathematics.

K-Theory. Another technique, which is in many ways very similar to homology theory, has had wide applications, and permeates many parts of mathematics, was of later origin. It did not emerge until the middle of the 20th century, although it is something that had its roots much further back as well. It is called "K-theory", and it is actually closely related to representation theory. Representation theory of, say, finite groups goes back to the last century, but its modern form, K-theory, is of more recent origin. K-theory can also be thought of in the following way: it is the attempt to take matrix theory, where matrices do not commute under multiplication, and try to construct Abelian or linear invariants of matrices. Traces and dimensions and determinants are Abelian invariants of matrix theory and K-theory is a systematic way of trying to deal with them; it is sometimes called "stable linear algebra". The idea is that if you have large matrices, then a matrix A and a matrix B that do not commute will commute if you put them in orthogonal positions in different blocks. Since in a big space you can move things around, then in some approximate way you might think this is going to be good enough to give you some information, and that is the basis of K-theory as a technique. It is analogous to homology theory, in that both try to extract linear information out of complicated non-linear situations.

In algebraic geometry, K-theory was first introduced with remarkable success by Grothendieck, in close relation to the story we just discussed a moment ago involving sheaf theory, and in connection with his work on the Riemann-Roch theorem.

In topology, Hirzebruch and I copied these ideas and applied them in a purely topological context. In a sense, while Grothendieck's work is related to Hilbert's work on syzygies, our work was more related to the Riemann-Poincaré work on homology, us-

ing continuous functions as opposed to polynomials. It also played a role in the index theory of linear elliptic partial differential equations.

In a different direction, the algebraic side of the story, with potential application to number theory, was then developed by Milnor, Quillen, and others, and has led to many interesting questions.

In functional analysis, the work of many people, including Kasparov, extended the continuous K-theory to the situation of non-commutative C^*-algebras. The continuous functions on a space form a commutative algebra under multiplication, but non-commutative analogues of those arise in other situations, and functional analysis turns out to be a very natural home for these kinds of questions.

So K-theory is another area where a whole range of different parts of mathematics lends itself to this rather simple formalism, although in each case there are quite difficult technical questions specific to that area, which connect up with other parts of the subject. It is not a uniform tool; it is more a uniform framework, with analogies and similarities between one part and the other.

Much of this work has also been extended by Alain Connes to "non-commutative differential geometry".

Interestingly enough, very recently, Witten in working on string theory (the latest ideas in fundamental physics) has identified very interesting ways in which K-theory appears to provide a natural home for what are called "conserved quantities". Whereas in the past it was thought that homology theory was the natural framework for them, it now seems that K-theory provides a better answer.

Lie Groups. Another unifying concept that is not just a technique is that of Lie groups. Now Lie groups, by which we mean fundamentally the orthogonal, unitary, and symplectic groups, together with some exceptional groups, have played a very important part in the history of 20th-century mathematics. Again, they date from the 19th century. Sophus Lie was a 19th-century Norwegian mathematician, and he, Felix Klein, and others pushed "the theory of continuous groups", as it was called. Originally, for Klein, this was a way of trying to unify the different kinds of geometry: Euclidean geometry and non-Euclidean geometry. Although this subject started in the 19th century, it really took off in the 20th century. The 20th century has been very heavily dominated by the theory of Lie groups as a sort of unifying framework in which to study many different questions.

I did mention the role in geometry of the ideas of Klein. For Klein, geometries were spaces that were homogeneous, where you could move things around without distortion, and so they were determined by an associated isometry group. The Euclidean group gave you Euclidean geometry; hyperbolic geometry came from another Lie group. So each homogeneous geometry corresponded to a different Lie group. But later on, following up on Riemann's work on geometry, people were more concerned with geometries that were not homogeneous, where the curvature varied from place to place and there were no global symmetries of space. Nevertheless, Lie groups still played an important role because they come in at the infinitesimal level, since in the tangent space we have Euclidean coordinates. Therefore, in the tangent space, infinitesimally, Lie group theory reappears, but because you have to compare different points in different places, you have to move things around in some way to handle the different Lie groups. That was the theory developed by Elie Cartan, the basis of modern differential geometry, and it was also the framework that was essential to Einstein's theory of relativity. Einstein's theory, of course, gave a big boost to the whole development of differential geometry.

Moving on into the 20th century, the global aspect, which I mentioned before, involved Lie groups and differential geometry at the global level. A major development, characterized by the work of Borel and Hirzebruch, gave information about what are called "characteristic classes". These are topological invariants combining the three key parts: the Lie groups, the differential geometry and the topology, and of course, the algebra associated with the group itself.

In a more analytical direction, we get what is now called non-commutative harmonic analysis. This is the generalization of Fourier theory, where the Fourier series or Fourier integrals correspond essentially to the commutative Lie groups of the circle and the straight line. When you replace these by more complicated Lie groups, then we get a very beautiful, elaborate theory that combines representation theory of Lie groups and analysis. This was essentially the lifework of Harish-Chandra.

In number theory the whole "Langlands program", as it is called, which is closely related also to Harish-Chandra's theory, takes place within the theory of Lie groups. For every Lie group, you have the associated number theory and the Langlands program, which has been carried out to some extent. It has influenced a large part of the work in algebraic number theory in the second half of this century. The study of modular forms fits into this part of the story, including Andrew Wiles' work on Fermat's Last Theorem.

One might think that Lie groups are particularly significant only in geometrical contexts, because of the need for continuous variation, but the analogues of Lie groups over finite fields give finite groups, and most finite groups arise in that way. Therefore the techniques of some parts of Lie theory apply even in a discrete situation for finite fields or for local fields. There is a lot of work that is pure algebra; for example, work with which George Lusztig's name is associated, where representation theory of such finite groups is studied and where many of the techniques that I have mentioned before have their counterparts.

7. FINITE GROUPS. This brings us to finite groups, and that reminds me: the classification of finite simple groups is something where I have to make an admission. Some years ago I was interviewed, when the finite simple group story was just about finished, and I was asked what I thought about it. I was rash enough to say I did not think it was so important. My reason was that the classification of finite simple groups told us that most simple groups were the ones we knew, and there was a list of a few exceptions. In some sense that closed the field, it did not open things up. When things get closed down instead of getting opened up, I do not get so excited, but of course a lot of my friends who work in this area were very, very cross. I had to wear a sort of bulletproof vest after that!

There is one saving grace. I did actually make the point that in the list of the so-called "sporadic groups", the biggest was given the name of the "Monster". I think the discovery of this Monster alone is the most exciting output of the classification. It turns out that the Monster is an extremely interesting animal and it is still being understood now. It has unexpected connections with large parts of other parts of mathematics, with elliptic modular functions, and even with theoretical physics and quantum field theory. This was an interesting by-product of the classification. Classifications by themselves, as I say, close the door; but the Monster opened up a door.

8. IMPACT OF PHYSICS. Let me move on now to a different theme, which is the impact of physics. Throughout history, physics has had a long association with mathematics, and large parts of mathematics, calculus, for example, were developed in order to solve problems in physics. In the middle of the 20th century this perhaps had be-

come less evident, with most of pure mathematics progressing very well independently of physics, but in the last quarter of this century things have changed dramatically. Let me try to review briefly the interaction of physics with mathematics, and in particular with geometry.

In the 19th century, Hamilton developed classical mechanics, introducing what is now called the Hamiltonian formalism. Classical mechanics has led to what we call "symplectic geometry". It is a branch of geometry that could have been studied much earlier, but in fact has not been studied seriously until the last two decades. It turns out to be a very rich part of geometry. Geometry, in the sense I am using the word here, has three branches: Riemannian geometry, complex geometry, and symplectic geometry, corresponding to the three types of Lie groups. Symplectic geometry is the most recent of these and in some ways possibly the most interesting, and certainly one with extremely close relations to physics, because of its historical origins in connection with Hamiltonian mechanics and more recently with quantum mechanics. Now, Maxwell's equations, which I mentioned before, the fundamental linear equations of electromagnetism, were the motivation for Hodge's work on harmonic forms, and the application to algebraic geometry. This turned out to be an enormously fruitful theory, which has underpinned much of the work in geometry since the 1930s.

I have already mentioned general relativity and Einstein's work. Quantum mechanics, of course, provided an enormous input. Not only in the commutation relations, but more significantly in the emphasis on Hilbert space and spectral theory.

In a more concrete and obvious way, crystallography in its classical form was concerned with the symmetries of crystal structures. The finite symmetry groups that can take place around points were studied in the first instance because of their applications to crystallography. In this century, the deeper applications of group theory have turned out to have relations to physics. The elementary particles of which matter is supposed to be built appear to have hidden symmetries at the very smallest level, where there are some Lie groups lurking around that you cannot see, but the symmetries of these become manifest when you study the actual behavior of the particles. So you postulate a model in which symmetry is an essential ingredient and the different theories that are now prevalent have certain basic Lie groups such as $SU(2)$ and $SU(3)$ built into them as primordial symmetry groups. So these Lie groups appear as building blocks of matter.

Nor are compact Lie groups the only ones that appear. Certain non-compact Lie groups, such as the Lorentz group, appear in physics. It was physicists who first started the study of the representation theory of non-compact Lie groups. These are representations that have to take place in Hilbert space because, for compact groups, the irreducible representations are finite dimensional, but non-compact groups require infinite dimensions, and it was physicists who first realized this.

In the last quarter of the 20th century, the one we have just been finishing, there has been a tremendous incursion of new ideas from physics into mathematics. This is perhaps one of the most remarkable stories of the whole century. It requires perhaps a whole lecture on its own but, basically, quantum field theory and string theory have been applied in remarkable ways to get new results, ideas, and techniques in many parts of mathematics. By this I mean that the physicists have been able to predict that certain things will be true in mathematics based on their understanding of the physical theory. Of course, that is not a rigorous proof, but it is backed by a very powerful amount of intuition, special cases, and analogies. These results predicted by the physicists have time and again been checked by the mathematicians and found to be fundamentally correct, even though it is quite hard to produce proofs and many of them have not yet been fully proved.

So there has been a tremendous input over the last 25 years in this direction. The results are extremely detailed. It is not just that the physicists said, "this is the sort of thing that should be true." They said, "here is the precise formula and here are the first ten cases" (involving numbers with more than 12 digits). They give you exact answers to complicated problems, not the kind of thing you can guess; things you need to have machinery to calculate. Quantum field theory has provided a remarkable tool, which is very difficult to understand mathematically but has had an unexpected bonus in terms of applications. This has really been the exciting story of the last 25 years.

Here are some of the ingredients: Simon Donaldson's work on 4-dimensional manifolds; Vaughan Jones' work on knot invariants; mirror symmetry, quantum groups; and I mentioned the Monster just for good measure.

What is this subject all about? As I mentioned before, the 20th century saw a shift in the number of dimensions ending up with an infinite number. Physicists have gone beyond that. In quantum field theory they are really trying to make a very detailed study of infinite-dimensional space in depth. The infinite-dimensional spaces they deal with are typically function spaces of various kinds. They are very complicated, not only because they are infinite-dimensional, but they have complicated algebra and geometry and topology as well, and there are large Lie groups around, infinite-dimensional Lie groups. So, just as large parts of 20th-century mathematics were concerned with the development of geometry, topology, algebra, and analysis on finite-dimensional Lie groups and manifolds, this part of physics is concerned with the analogous treatments in infinite dimensions, and of course it is a vastly different story, but it has enormous payoffs.

Let me explain this in a bit more detail. Quantum field theories take place in space and time; and space is really meant to be three-dimensional but there are simplified models where you take one dimension. In one-dimensional space and one-dimensional time, typically the things that physicists meet are, mathematically speaking, groups such as the diffeomorphisms of the circle or the group of differentiable maps from the circle into a compact Lie group. These are two very fundamental examples of infinite-dimensional Lie groups that turn up in quantum field theories in these dimensions, and they are quite reasonable mathematical objects that have been studied by mathematicians for some time.

In such $1 + 1$ dimensional theories one can take space-time to be a Riemann surface, and this leads to new results. For example, the moduli space of Riemann surfaces of a given genus is a classical object going back to the last century. Quantum field theory has led to new results about the cohomology of these moduli spaces. Another, rather similar, moduli space is the moduli space of flat G-bundles over a Riemann surface of genus g. These spaces are very interesting, and quantum field theory gives precise results about them. In particular, there are beautiful formulas for the volumes, which involve values of zeta functions.

Another application is concerned with counting curves. If you look at plane algebraic curves of a given degree of a given type, and you want to know how many of them, for example, pass through so many points, you get into enumerative problems of algebraic geometry, problems that would have been classical in the last century. These are very hard. They have been solved by modern machinery called "quantum cohomology", which is all part of the story coming from quantum field theory, or you can look at more difficult questions about curves not in the plane, but curves lying on curved varieties. One gets another beautiful story with explicit results going by the name of mirror symmetry. All this comes from quantum field theory in $1 + 1$ dimensions.

If we move up one dimension, where we have 2-space and 1-time, this is where Vaughan Jones' theory of invariants of knots comes in. This has had an elegant explanation or interpretation in quantum-field-theory terms.

Also coming out of this is what are called "quantum groups". Now the nicest thing about quantum groups is their name. They are definitely not groups! If you were to ask me for the definition of a quantum group, I would need another half hour. They are complicated objects, but there is no question that they have a deep relationship with quantum theory. They emerged out of the physics, and they are being applied by hard-nosed algebraists who actually use them for definite computations.

If we move up one step further, to fully four-dimensional theory (three-plus-one dimension), that is where Donaldson's theory of four-dimensional manifolds fits in and where quantum field theory has had a major impact. In particular, it led Seiberg and Witten to produce their alternative theory, which is based on physical intuition and gives marvellous results mathematically as well. All of these are particular examples. There are many more.

Then there is string theory and this is already passé! *M*-theory is what we should talk about now, and that is a rich theory, again with a large number of mathematical aspects to it. Results coming out of it are still being digested and will keep mathematicians busy for a long time to come.

9. HISTORICAL SUMMARY.

Let me just try to make a quick summary. Let me look at the history in a nutshell: what has happened to mathematics? I will rather glibly just put the 18th and 19th centuries together, as the era of what you might call classical mathematics, the era we associate with Euler and Gauss, where all the great classical mathematics was worked out and developed. You might have thought that would almost be the end of mathematics, but the 20th century has, on the contrary, been very productive indeed and this is what I have been talking about.

The 20th century can be divided roughly into two halves. I would think the first half has been dominated by what I call the "era of specialization", the era in which Hilbert's approach, of trying to formalize things and define them carefully and then follow through on what you can do in each field, was very influential. As I said, Bourbaki's name is associated with this trend, where people focused attention on what you could get within particular algebraic or other systems at a given time. The second half of the 20th century has been much more what I would call the "era of unification", where borders are crossed, techniques have been moved from one field into the other, and things have become hybridized to an enormous extent. I think this is an oversimplification, but I think it does briefly summarize some of the aspects that you can see in 20th-century mathematics.

What about the 21st century? I have said the 21st century might be the era of quantum mathematics or, if you like, of infinite-dimensional mathematics. What could this mean? Quantum mathematics could mean, if we get that far, understanding properly the analysis, geometry, topology, algebra of various non-linear function spaces, and by "understanding properly" I mean understanding it in such a way as to get quite rigorous proofs of all the beautiful things the physicists have been speculating about.

One should say that, if you go at infinite dimensions in a naive way and ask naive questions, you usually get the wrong answers, or the answers are dull. Physical application, insight, and motivation have enabled physicists to ask intelligent questions about infinite dimensions and to do very subtle things where sensible answers do come out, and therefore doing infinite-dimensional analysis in this way is by no means a simple task. You have to go about it in the right way. We have a lot of clues. The map is laid out: this is what should be done, but it is long way to go yet.

What else might happen in the 21st century? I would like to emphasize Connes' non-commutative differential geometry. Alain Connes has this rather magnificent unified theory. Again it combines everything. It combines analysis, algebra, geometry, topology, physics, and number theory, all of which contribute to parts of it. It is a framework that enables us to do what differential geometers normally do, including its relationship with topology, in the context of non-commutative analysis. There are good reasons for wanting to do this, applications (potential or otherwise) in number theory, geometry, discrete groups, and so on, and in physics. An interesting link with physics is just being worked out. How far this will go, what it will achieve, remains to be seen. It certainly is something that I expect will be significantly developed in the first decade at least of the next century, and it is possible it could have a link with the as-yet-undeveloped (rigorous) quantum field theory.

Moving in another direction, there is what is called "arithmetic geometry" or Arakelov geometry, which tries to unify as much as possible algebraic geometry and parts of number theory. It is a very successful theory. It has made a nice start but has a long way to go. Who knows?

Of course, all of these have strands in common. I expect physics to have its impact spread all the way through, even to number theory: Andrew Wiles disagrees and only time will tell.

These are the strands that I can see emerging over the next decade, but there is what I call a joker in the pack: going down to lower-dimensional geometry. Alongside all the infinite-dimensional fancy stuff, low-dimensional geometry is an embarrassment. In many ways the dimensions where we started, where our ancestors started, remain something of an enigma. Dimensions 2, 3, and 4 are what we call "low". For example, the work of Thurston in three-dimensional geometry aims at a classification of geometries one can put on three-dimensional manifolds. This is much deeper than the two-dimensional theory. The Thurston program is by no means completed yet, and completing that program certainly should be a major challenge.

The other remarkable story in three dimensions is the work of Vaughan Jones with ideas essentially coming from physics. This gives us more information about three dimensions, which is almost orthogonal to the information contained in the Thurston program. How to link those two sides of the story together remains an enormous challenge, but there are recent hints of a possible bridge. So this whole area, still in low dimensions, has its links to physics, but it remains very mysterious indeed.

Finally, I should like to mention that in physics what emerges very prominently are "dualities". These dualities, broadly speaking, arise when a quantum theory has two different realizations as a classical theory. A simple example is the duality between position and momentum in classical mechanics. This replaces a space by its dual space, and in linear theories that duality is just the Fourier transform. But in non-linear theories, how you replace a Fourier transform is one of the big challenges. Large parts of mathematics are concerned with how to generalize dualities in non-linear situations. Physicists seem to be able to do so in a remarkable way in their string theories and in M-theory. They produce example after example of marvellous dualities that in some broad sense are infinite-dimensional non-linear versions of Fourier transforms and they seem to work. But understanding those non-linear dualities does seem to be one of the big challenges of the next century as well.

I think I will stop there. There is plenty of work, and it is very nice for an old man like me to talk to a lot of young people like you; to be able to say to you: there is plenty of work for you in the next century!

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