Elliptic Operators and Topology: Lecture 10

ΨDOs

Two more properties of ΨDOs:

(a) diffeomorphism invariant

(b) L^2-estimates.

(a): **Theorem** Let χ be a diffeo. $\mathbb{R}^n \circlearrowleft$ and let $Lu = u \circ \chi^{-1}$.
If P is a ΨDO then so is $L^{-1} P L = \tilde{P}$.
(Furthermore, there is an asymptotic formula for the symbol of $\tilde{P}$.)

Pf: (sketch) $\tilde{k}(x,y) = k(\chi(x), \chi(y)) \underbrace{j(y)}_{\text{Jacobian term}}$

so $\int e^{i(\chi(x)-\chi(y))\cdot\xi} p(x,\xi) j(y)\, d\xi$.

Note: χ a diffeo $\Rightarrow$ χ is Lipschitz $\Rightarrow$ $\dfrac{\chi(x)-\chi(y)}{x-y}$ is smooth on $\mathbb{R}^n \times \mathbb{R}^n$.

So $\tilde{k}(x,y) = \int e^{i(x-y)\cdot \boxed{M(x,y)\xi} = \eta} p(x,\xi) j(y)\, d\xi$

where $(M(x,y))_{ij} = \dfrac{(\chi(x)-\chi(y))_i}{(x-y)_j} \in \mathrm{End}(\mathbb{R}^n)$. Changing co-ordinates by

$\eta = M(x,y)\xi$ gives

$$\tilde{k}(x,y) = \int e^{i(x-y)\cdot\eta} \underbrace{p(x, M(x,y)^{-1}\eta)\, j(y) \det(M(x,y))^{-1}}_{= a(x,y,\eta)} d\eta$$

and so we can define ΨDOs on manifolds. These are operators

$T: C^\infty(M) \circlearrowleft$ such that

(a) T is pseudo-local

(b) in local co-ordinates, T is a ΨDO on $\mathbb{R}^n$.

The principal symbol is well-defined in $\underbrace{S^m(T^*M)}_{\text{Symbol space of order} \leq m}/S^{m-1}(T^*M)$

$= \{\sigma(x,\xi) \mid |\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C(1+|\xi|)^{m-|\beta|}\}$.

Thm On a compact manifold, a ΨDO P of order $\leq m$ satisfies

$$\|Pu\|_{k-m} \leq C_{k,m} \|u\|_m$$

Pf it is sufficient this for $m=0$ (we can use the Laplacian Δ on $\mathbb{T}^2$ to reduce to order 0 case, as $\widehat{(I-\Delta)u}(\xi) = (1+|\xi|^2)\hat{u}(\xi)$, hence conjugating by $I-\Delta$ allows us to reduce or add terms). We want to synthesise P from constant coeff. operators (i.e. ~~your~~ their symbols $C = C(\xi)$ are independent of x).

$\|C\|_{B(L^2)} = \sup_\xi |C(\xi)|$ by Plancherel $(\|u\|_{L^2} = \|\hat{u}\|_{L^2})$.

Let $\hat{p}(\eta,\xi)$ be the x-Fourier transform of $p(x,\xi)$, so

$$p(x,\xi) = \int \hat{p}(\eta,\xi)\, e^{ix\cdot\eta}\, d\eta.$$

Let P_η be the constant coefficient operator with symbol $\xi \mapsto \hat{p}(\eta,\xi)$.

Then $Pu(x) = \int P_\eta u(x) e^{ix\cdot\eta}\, d\eta$. So

$$\|Pu\|_{L^2} \leq \int \|P_\eta\|_{L^2} \|u\|_{L^2}\, d\eta = \|u\|_{L^2} \int \underbrace{\sup_\xi |\hat{p}(\eta,\xi)|}_{\leq (1+|\eta|)^{-N}\ \forall N} d\eta$$

for some $C>0$.

$$\Rightarrow \|Pu\|_{L^2} \leq C\|u\|_{L^2}$$

Lefschetz Theorem

Elliptic complex

$$\begin{tikzcd} C^\infty(E_0) \arrow[r, "d"] \arrow[loop below, "T_0"] & C^\infty(E_1) \arrow[r, "d"] \arrow[loop below, "T_1"] & C^\infty(E_2) \arrow[r] \arrow[loop below, "T_2"] & \cdots \end{tikzcd}$$

and a geometric endomorphism $T_i \in \mathrm{End}\left(C^\infty(E_i)\right)$ (e.g. $f^*: \Omega^*(M) \circlearrowleft$).

Then set $L(T) \triangleq \sum_i (-1)^i \,\mathrm{Tr}\left(H^i(T)\right)$, where $H^i(T): H^i(C^\infty(E_*)) \to H^i(C^\infty(E_*))$ is the operator induced on cohomology (well-defined, as $T_{i+1}\circ d = d\circ T_i$).

The fixed point theorem states that $L(T) = \sum_{f(p)=p} \nu(p)$.

Basic Linear Algebra Fact

Suppose we have a complex of vector spaces (fin. dim.)

$$\begin{tikzcd} V_0 \arrow[r, "d"] \arrow[d, "T_0"'] & V_1 \arrow[r, "d"] \arrow[d, "T_1"] & V_2 \arrow[r] \arrow[d, "T_2"] & \cdots \\ V_0 \arrow[r, "d"] & V_1 \arrow[r, "d"] & V_2 \arrow[r] & \cdots \end{tikzcd}$$

T_i endomorphisms

__Lemma__ $\sum_i (-1)^i \,\mathrm{Tr}\left(H^i(T)\right) = \sum (-1)^i \,\mathrm{Tr}(T_i)$

(e.g. $T_i = \mathrm{Id}_{V_i}$, $V_i = \Omega^i(M) \Rightarrow \chi(M) = \sum_i (-1)^i \,\mathrm{rk}\left(H^i(M)\right) = \sum_i (-1)^i \,\mathrm{rk}\left(\Omega^i(M)\right)$)

Pf introduce inner products on the $V_i \Rightarrow$ an adjoint d^* exists.

We can now do ~~hodge~~ Hodge theory with

$$\Delta = dd^* + d^*d = (d+d^*)^2.$$

Define harmonic subspaces $H^i(V,d) \triangleq \ker(\Delta_i)$. Let $e^{-t\Delta}$ be the corresponding heat operator, and note that $e^{-t\Delta} \to \mathrm{Id}$ as $t \to 0$ and $e^{-t\Delta} \to P$ as $t \to \infty$, where P = projection onto harmonic forms.

To see why this is, let φ be an eigenfunction of Δ, so
$\Delta\varphi = \lambda\varphi$ (for $\lambda \geq 0$). So

$$e^{-t\Delta}\varphi = e^{-t\lambda}\varphi \longrightarrow \begin{cases} \varphi & \lambda = 0 \\ 0 & \lambda > 0 \end{cases}$$

. The spectrum of Δ is ≥ 0 as $(\Delta\varphi, \varphi) = |(d+d^*)\varphi|^2 \geq 0$.

So, if $u = u_H + u_\perp$, for $u_H \in \ker(\Delta)$ and $u_\perp$ orthogonal to $\ker(\Delta)$, then $e^{-t\Delta} u \longrightarrow u_H$ as $t \to \infty$.

<u>Claim:</u> $f(t) = \sum_i (-1)^i \, \mathrm{Tr}(T_i e^{-t\Delta_i})$ is independent of t.

<u>Pf:</u> consider $f'(t) = \sum_i (-1)^i \, \mathrm{Tr}(T_i e^{-t\Delta_i} \Delta_i)$. Now,

$$\begin{aligned} \mathrm{Tr}(T_i e^{-t\Delta_i} d^* d) &= \mathrm{Tr}(T_i e^{-t\Delta_i} d^*_{i+1} d_i) \\ &= \mathrm{Tr}(d_i T_i e^{-t\Delta_i} d^*_{i+1}) \quad \text{by the symmetry of the trace} \\ &= \mathrm{Tr}(T_{i+1} d_i e^{-t\Delta_i} d^*_{i+1}) \\ &= \mathrm{Tr}(T_{i+1} e^{-t\Delta_i} d_i d^*_{i+1}), \end{aligned}$$

where we used $d_i \Delta_i = d_i (d_{i-1} d_i^* + d^*_{i+1} d_i) = d_i d^*_{i+1} d_i = (d^*_{i+1} d_{i+1} + d_i d^*_{i+1}) d_i$
$= \Delta_{i+1} d_i$.

So the terms $(-1)^i \mathrm{Tr}(T_i e^{-t\Delta_i} \Delta_i) = (-1)^i \left[\mathrm{Tr}(T_i e^{-t\Delta_i} d_{i-1} d_i^*) + \mathrm{Tr}(T_{i+1} e^{-t\Delta_{i+1}} d_i d^*_{i+1}) \right]$

cancel off in pairs, hence $f'(t) = 0$.

So $f(\infty) = f(0)$, i.e.

$$\sum_i (-1)^i \, \mathrm{Tr}(H^i(T)) = \sum_i (-1)^i \, \mathrm{Tr}(T_i).$$

This argument is set up to generalize to elliptic complexes.

Extend the notion of <u>trace</u>:

(a) <u>finite rank</u> operators

$\Downarrow$

(b) <u>smoothing</u> operators

Consider $Au(x) = \int_M k(x,y)u(y)\,d\lambda(y)$ for $k \in C^\infty(M\times M)$.

Define $\mathrm{Tr}(A) = \int_M k(x,x)\,d\lambda(x)$.

By Fubini (M is compact, otherwise need k to have compact support),

$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ as

$$\mathrm{Tr}(AB) = \int_M\int_M k_A(x,y)\,k_B(y,x)\,d\lambda(y)\,d\lambda(x)$$

$$= \int_M\int_M k_B(y,x)\,k_A(x,y)\,d\lambda(x)\,d\lambda(y)$$

$$= \mathrm{Tr}(BA).$$

This is analogous to the usual definition of trace by $\xi \mapsto \langle \xi, a\rangle b$, but here we have $a(x)\,b(y)$.

Recall the function $f(t) = \sum_i (-1)^i\,\mathrm{Tr}(T_i e^{-t\Delta_i})$. This is fine for $t\to\infty$, but at $t=0$ we seem to get $\infty-\infty$ terms. So it will require some more analysis. Consider the diagonal map

$$\mathrm{diag}: M \longrightarrow M\times M$$
$$x \longmapsto (x,x).$$

If k is a distribution on $M\times M$, we wish to pull it back via diag. Generally, this is impossible (e.g. δ on $\mathbb{R}$ pulled back by the constant map $x\mapsto 0$). However, we can work around this if the distribution k is the kernel of a ΨDO.

Consider the graph map

$$\Gamma_f : M \longrightarrow M \times M \,.$$
$$x \longmapsto (x, f(x))$$

Then $\int_M k(x, f(x))\, d\lambda(x)$ makes sense, and we will use this to define the trace at $t = 0$.