Elliptic Operators and Topology

Lecture 7, 2014. 9. 16

- Sobolev Spaces $W^k(M)$.

  – "space of functions (or section) whose first k derivative belongs to L^2".

  Or, Fourier coefficients a_γ satisfies $\sum_{\gamma \in \mathbb{Z}^n} |a_\gamma|^2 (1+|\gamma|^2)^k < \infty$.

  Note: For any polynomial p of degree $\le k$, $\exists C$, s.t

$$|p(\gamma)| \le C\,(1+|\gamma|^2)^{k/2}.$$

- Distribution Theory.

  M compact. $C^\infty(M)$ is space of smooth functions on M.

  $(C^\infty(M))'$: dual space of $C^\infty(M)$ = space of distributions on M.

  An operator T on $C^\infty(M)$, induces an operator T^* on $(C^\infty(M))'$.

Example. $H(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0. \end{cases}$

$$\Rightarrow H'(x) = \delta(x), \quad \text{or} \quad \int_{\mathbb{R}} H'(x) f(x)\,dx = f(0), \quad \forall f \in C_c^\infty(\mathbb{R}).$$

$$C^\infty := W^\infty \subseteq \cdots \subseteq W^0 \subseteq W^{-1} \subseteq \cdots \subseteq \underset{\substack{\parallel \\ \text{space of distributions}}}{W^{-\infty}} = \bigcup_{k \in \mathbb{Z}} W^k$$

Note. See more about functional analysis on the notes, which will be posted.

- Theorem. Suppose D is elliptic of order m, then there is Q (order $-m$), which is an inverse to D, modulo smooth operators (of order $-\infty$).

$$\|Qu\|_{W^{k+m}} \le C_k \|u\|_{W^k}.$$

Corollary 2: $D: W^m \longrightarrow L^2$ has closed range.

Proof: See last lecture (Lecture notes 6).

- Smoothing Operators.

(A) – continuous linear operator from $W^{-\infty}$ to W^{∞}.

(B) – $Tu(x) = \int_M k(x,y)\, u(y)\, d\lambda(y)$,

where $k(x,y) \in C^\infty(M \times M)$ is called "kernel" of T, and $\lambda(y)$ is any smooth lebesgue measure on M. Note that u can be a distribution.

Theorem. (A) and (B) are the same.

proof. (A) $\Rightarrow$ (B).

Note that, for any $z \in M$, there is a distribution δ_z, such that

$$\int_M v(x)\, \delta_z(x)\, d\lambda(x) = v(z).$$

Then define T: $\quad T\delta_z(x) = K(x,z)$, which is smooth on x and z.

Note: The map $M \longrightarrow \mathcal{D}'(M)$

$z \longmapsto \delta_z$ is smoothly dependent on z.

(B) $\Rightarrow$ (A) Let u be a distribution in (B).

- Schwarz Kernel Theorem.

$$u(x) = \int_{\mathbb{R}} \delta(x-y)\, u(y)\, dy, \quad \text{but } \delta(x-y) \text{ is not smooth function.}$$

- Let M be compact, T a linear map: {smooth functions} $\longrightarrow$ {distributions}, then T can be represented by

$$Tu(x) = \int_M k(x,y)\, u(y)\, d\lambda(y),$$

in the sense that

$$\langle Tu(x), v(x)\rangle = \int_M \int_M k(x,y)\, u(y)\, v(x)\, d\lambda(y)\, d\lambda(x).$$

- Oscillatory Integrals.

Fourier inversion formula:

$$\hat{u}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\cdot\xi} u(y)\, dy$$

$$u(x) = \int_{\mathbb{R}} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi.$$

$$\Rightarrow u(x) = \int_{\mathbb{R}} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ix\cdot\xi} \cdot e^{-iy\cdot\xi} u(y)\, dy$$

$$= \int_{\mathbb{R}} \left(\frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x-y)\cdot\xi}\, d\xi \right) u(y)\, dy$$

So in the distribution sense,

$$\delta(x-y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x-y)\cdot\xi}\, d\xi$$

Question ?: How to make sense of an "oscillatory integral"?

Define $k(x,y) = \int e^{i(x-y)\cdot\xi} a(x,y,\xi)\, d\xi$,

$\nwarrow$ amplitude.

where amplitude a is required to satisfy certain regularity conditions.

From $\hat{u}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\cdot\xi} u(x)\, dx$, then

$$\xi\hat{u}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} \xi e^{-ix\cdot\xi} u(x)\, dx$$

$$= \frac{i}{2\pi}\int_{\mathbb{R}} \frac{d}{dx}(e^{-ix\cdot\xi})\, u(x)\, dx$$

$$= -\frac{i}{2\pi}\int_{\mathbb{R}} e^{-ix\cdot\xi} u'(x)\, dx$$

$$= -i\,\widehat{u'}(\xi) \quad , \text{ for } u \in C_c^\infty(\mathbb{R}).$$

Similarly, $\xi^n \hat{u}(\xi) = (-i)^n \widehat{u^{(n)}}(\xi)$.

- Asymptotic Expansions.

Suppose $f \in C^\infty(\mathbb{R})$, its Taylor series is $\sum_{n\geq 0} a_n x^n$, where $a_n = \frac{f^{(n)}(0)}{n!}$.

Facts: - Taylor series need not to converge outside 0.

- Even if it does, may not be f.

Nevertheless, it is an asymptotic series in the sense that, for any $N>0$,

$$\left| f(x) - \sum_{k=0}^{N-1} a_k x^k \right| = O(x^N) \leq C_N |x|^N.$$

Theorem (Borel). Given any sequence $\{a_n\}$ of reals, there exists $f \in C^\infty(\mathbb{R})$ with those Taylor coefficients.