Elliptic Operators and Topology: Lecture 8

Recall: last time we spoke about
- the Schwarz kernels theorem
- oscillatory integrals
- asymptotic series

Today we will work locally on $\mathbb{R}^n$.

Def: let $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. Then a is an amplitude of order $\leq m$ if for any indices (α, β, γ) and $K \subseteq \mathbb{R}^n \times \mathbb{R}^n$ compact, $\exists\ C = C(\alpha, \beta, \gamma, K)$ s.t.
$$\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x,y,\xi) \right| \leq C (1 + |\xi|)^{m - |\gamma|} \quad \forall (x,y) \in K.$$

A symbol is an amplitude which is independent of y.

Examples: (i) if a is a polynomial in ξ whose coefficients are functions of (x,y) then ~~its an~~ it is an amplitude of order = ~~order~~ degree of a in ξ.

(ii) $(1 + |\xi|^2)^{-1}$ has order -2, as $\frac{d}{d\xi} \frac{1}{1+\xi^2} = \frac{-2\xi}{(1+\xi^2)^2}$ and so on.

(iii) amplitudes form a filtered algebra under pointwise multiplication, by the product rule.

(iv) $\partial_x, \partial_y, \partial_\xi$ map amplitudes to amplitudes.

Def: a pseudo-differential operator (denote these by Ψ-DO) on $\mathbb{R}^n$ is a linear map $C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ defined by a Schwarz kernel
$$k(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi)\, d\xi$$

where a is an amplitude.

So $Tu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,y,\xi)\, u(y)\, d\xi\, dy$ gives such a map ~~$T: C_c^\infty(\mathbb{R}^n)$~~

$T: C_c^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

What must we do to make sense of this? If a has order $< -n$, then the integrand lies in $L^1(\mathbb{R}^n)$ $\forall\, x \in \mathbb{R}^n$ and so it converges (by the dominated convergence theorem and Fubini), hence it defines a continuous function on $\mathbb{R}^n$.

If a ~~is~~ is an amplitude of order $\leq -n-k$, then we ~~do~~ can write

$$\cancel{T(x+he_j)} \quad \frac{Tu(x+he_j) - Tu(x)}{h} \cancel{= \iint \left(\frac{1}{2\pi}\right)^n u(y)\, e^{i(x-y)\cdot\xi}\left(e^{ih\cdot\xi_j} - 1\right)}$$

$$= \left(\frac{1}{2\pi}\right)^n \iint_{\mathbb{R}^n \times \mathbb{R}^n} u(y)\, e^{i(x-y)\cdot\xi} \left(\frac{e^{ih\xi_j}\, a(x+he_j, y, \xi) - a(x,y,\xi)}{h} \right) d\xi\, dy,$$

and as $\left| \frac{e^{ih\xi_j}\, a(x+he_j, y, \xi) - a(x,y,\xi)}{h} \right| \leq |\xi_j|\, |a(x+he_j, y, \xi)| + |\partial_{x_j} a(x+te_j, y, \xi)|$ for $|t| < |h|$, by the dominated convergence theorem we have that Tu is differentiable k times, i.e. $Tu \in C^k(\mathbb{R}^n)$.

Let $L = I + i \sum_{j=1}^{n} \xi_j \frac{\partial}{\partial y_j}$. Then $L\, e^{i(x-y)\cdot\xi} = (1 + |\xi|^2)\, e^{i(x-y)\cdot\xi}$, so

$$Tu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{L e^{i(x-y)\cdot\xi}}{1+|\xi|^2}\, a(x,y,\xi)\, u(y)\, d\xi\, dy$$

and integrating by parts yields (provided we can do this legitimately)

$$Tu(x) \text{ "=" } \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} \underbrace{\frac{e^{i(x-y)\cdot\xi}}{1+|\xi|^2}}_{\text{"improves" by 2 degrees}}\, L\big(\underbrace{a(x,y,\xi)\, u(y)}_{1 \text{ degree worse}}\big)\, d\xi\, dy$$

$\Rightarrow$ overall, an improvement.

Thm: the formal adjoint of a ΨDO is a ΨDO.

Pf: if T has kernel $k(x,y)$, then T^* has kernel $\overline{k(y,x)}$; i.e.

$$\langle Tu,v\rangle = \int_{\mathbb{R}^n\times\mathbb{R}^n} u(y)\,k(x,y)\,\overline{v(y)}\,dx\,dy = \int_{\mathbb{R}^n\times\mathbb{R}^n} u(y)\,\overline{\left(\overline{k(x,y)}\,v(y)\right)}\,dy\,dx$$

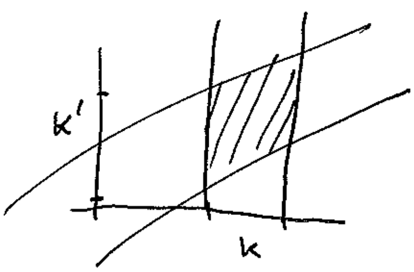
$$= \langle u, T^*v\rangle.$$

In our case, $k(x,y) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\,a(x,y,\xi)\,d\xi$

$$\Rightarrow \overline{k(y,x)} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\,\overline{a(y,x,\xi)}\,d\xi$$

and let $b(x,y,\xi) = \overline{a(y,x,\xi)}$.

Cor: if a is <u>properly supported</u> then the corresponding ΨDO acts on distributions.



Note: $a\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ is properly supported if $\forall$ $K\subseteq\mathbb{R}^n$ compact, $\exists$ $K'\subseteq\mathbb{R}^n$ compact s.t. $K\times\mathbb{R}^n\cap\operatorname{supp}(a)\subseteq\mathbb{R}^n\times K'$ and vice-versa (not necessarily same K, K').

Thm: ΨDO are <u>pseudo-local</u>, i.e. $k(x,y)$ is a smooth function away from the diagonal $\Delta=\{x=y\}$. Equivalently, $T\in$ ΨDO iff the singular support of Tu is contained in the singular support of u, i.e.

$$\operatorname{sing\,supp}(Tu)\subseteq\operatorname{sing\,supp}(u),$$

where $\operatorname{sing\,supp}(\varphi) = \{x \mid \varphi|_U \neq \psi \text{ for all } \psi\in C^\infty(U),\ U\ni x\}$.

Pf: (Suppose a is supported away from the diagonal.)

$$k(x,y) = \int e^{i(x-y)\cdot\xi}\,a(x,y,\xi)\,d\xi \quad\text{and let}\quad M = -i\sum_{j=1}^n (x_j-y_j)\frac{\partial}{\partial\xi_j}.$$

Then $M\left(e^{i(x-y)\cdot\xi}\right) = |x-y|^2 e^{i(x-y)\cdot\xi}$

$$\Rightarrow k(x,y) = \int \frac{M\left(e^{i(x-y)\cdot\xi}\right)}{|x-y|^2} a(x,y,\xi)\, d\xi, \quad \text{and integrate by parts}$$

$$\text{"}=\text{"} \int \underbrace{\frac{e^{i(x-y)\cdot\xi}}{|x-y|^2}}_{\text{good away from } \Delta} \underbrace{M\left(a(x,y,\xi)\right)}_{\text{1 degree better}} d\xi$$

$\Rightarrow k$ is smooth.

Doing this repeatedly will give a function that decreases as x or $y \to \infty$. So $k(x,y)$ is smooth in x & y. So the singular contribution only comes from values of a along the diagonal.

<u>Relation to differential operators:</u>

$$D = \sum_\alpha c_\alpha(x)\, \partial_x^\alpha$$

Recall: $\widehat{\partial_\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$

$$\Rightarrow Du(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(\sum_\alpha c_\alpha(x)(i\xi)^\alpha\right) \underbrace{\hat{u}(\xi)}_{= \int u(y) e^{-i\xi\cdot y} dy} d\xi$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-y)\cdot\xi} \left(\sum_\alpha \underbrace{c_\alpha(x)(i\xi)^\alpha}_{\substack{\text{amplitude} \\ \text{(no } y \text{ dependence)}}}\right) u(y)\, dy\, d\xi$$

$\Rightarrow D$ is a ΨDO with symbol $\sum_\alpha c_\alpha(x)(i\xi)^\alpha$, which agrees ~~so~~ with our previous definition of the symbol of a differential operator.

<u>Thm:</u> for any ΨDO T with amplitude $a(x,y,\xi)$ $\exists$ a symbol $p(x,\xi)$ ~~and~~ of the same order and defining the same operator, ~~up~~ up to smoothing terms.

Moreover, $p(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi)\right)\Big|_{y=x}$.