Elliptic Operators and Topology

Lecture 9, 2014. 9. 23

§. Asymptotics

Recall an amplitude $a(x,y,\xi)$ is of order m, if

$$\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a(x,y,\xi)\right| \le C_{\alpha,\beta,\gamma,K}(1+|\xi|)^{m-|\gamma|}.$$

Notation: $\mathcal{A}^m$ = class of amplitudes of order $\le m$.

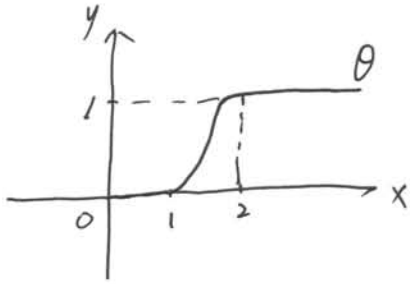
Def 1: Suppose $a \in \mathcal{A}^m$, $a_j \in \mathcal{A}^{m-j}$. We say $a \sim \sum_{j=0}^{\infty} a_j$, if

$$a - \sum_{j=0}^{N} a_j \in \mathcal{A}^{m-(N+1)}.$$

· Theorem (Borel Theorem for amplitudes).

Given any sequence $a_j \in \mathcal{A}^{m-j}$, there is an amplitude $a \in \mathcal{A}^m$, s.t

$$a \sim \sum_{j=0}^{\infty} a_j.$$



Proof. Let θ be the bump function.

Observation: Suppose $b \in \mathcal{A}^k$, consider

$$c(x,y,\xi,\lambda) = b(x,y,\xi)\,\theta\left(\lambda(1+|\xi|^2)^{\frac{1}{2}}\right) \in \mathcal{A}^{k} \subset \mathcal{A}^{k+1}$$

So In $\mathcal{A}^{k+1}$, as $\lambda \to 0$,

$$\sup_{x,y\in K,\ \xi\in\mathbb{R}^n} \frac{\left|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma c(x,y,\xi,\lambda)\right|}{(1+|\xi|)^{k+1-|\gamma|}} \longrightarrow 0.$$

Now we define $b_j(x,y,\xi) = a_j(x,y,\xi)\,\theta\left(\lambda_j(1+|\xi|^2)^{\frac{1}{2}}\right) \in \mathcal{A}^{m-j} \subset \mathcal{A}^{m-j+1}$.

and $a(x,y,\xi) = \sum_{j=0}^{\infty} b_j(x,y,\xi)$, where the sequence $\lambda_j \downarrow 0$ fast enough that

$$\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma b_j(x,y,\xi) \right| \le 2^{-j}\left(1+|\xi|\right)^{m-j+1-|\gamma|}.$$

So $a(x,y,\xi) = \sum_{j=0}^{\infty} a_j(x,y,\xi)\,\theta\left(\lambda_j (1+|\xi|^2)^{\frac{1}{2}}\right)$ converges.

2. $K(x,y) = \int e^{i(x-y)\cdot\xi} a(x,y,\xi)\, d\xi$, $a(x,y,\xi) \in A^m$.

- Theorem: Any pseudo-differential operator as above is equivalent to one with a symbol $p(x,\xi)$ (or one with symbol $q(y,\xi)$).

Idea of proof: From Taylor Theorem,

$$a(x,y,\xi) \sim \sum_\alpha \frac{(x-y)^\alpha}{\alpha!}\left[\partial_y^\alpha a(x,y,\xi)\right]\Big|_{y=x}$$

$$K(x,y) \sim \sum_\alpha \int e^{i(x-y)\cdot\xi} \frac{(x-y)^\alpha}{\alpha!}\left[\partial_y^\alpha a(x,y,\xi)\right]\Big|_{y=x} d\xi$$

$$= \int e^{i(x-y)\cdot\xi} \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\left[\partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi)\right]\Big|_{y=x} d\xi \quad \text{(integrating by parts).}$$

Define $p(x,\xi) = \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\left[\partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi)\right]\Big|_{y=x}$, then

$$K(x,y) \sim \int e^{i(x-y)\cdot\xi} p(x,\xi)\, d\xi$$

Note: One can make the proof rigorous using Definition 1.

- Corollary: The composite of two pseudo-differential operators is a pseudo-differential operator.

Proof. Assume T_1 is left quantized with symbol $p(x,\xi)$, T_2 is right quantized with symbol $q(z,\eta)$. Consider the symbol of $T_1 \circ T_2 =: T$.

$$k_1(x,y) = \int e^{i(x-y)\cdot\xi}\, p(x,\xi)\, d\xi$$

$$k_2(y,z) = \int e^{i(y-z)\cdot\eta}\, q(z,\eta)\, d\eta.$$

Then

$$K(x,z) = \int k_1(x,y)\, k_2(y,z)\, dy \qquad \cancel{(\text{the symbol of } T_1 \circ T_2)}.$$

$$= \iint \left(\int e^{i(\eta-\xi)\cdot y}\, dy \right) e^{ix\xi - iz\eta}\, p(x,\xi)\, q(z,\eta)\, d\xi\, d\eta$$

$$= \iint \delta(\eta-\xi)\, e^{ix\xi - iz\eta}\, p(x,\xi)\, q(z,\eta)\, d\xi\, d\eta$$

$$= \int e^{i(x-z)\cdot\xi} \underbrace{p(x,\xi)\, q(z,\xi)}_{a(x,z,\xi)}\, d\xi ,$$

where $a(x,z,\xi) = p(x,\xi)\, q(z,\xi)$ is a symbol of $T_1 \circ T_2$.

Note: One can make this proof more rigorous using Fourier transform.

- In particular, if T_1 and T_2 have symbols $p_1(x,\xi)$ and $p_2(x,\xi)$, then T has a left symbol $p(x,\xi)$ with the asymptotic expansion,

$$p(x,\xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} \left(\partial_\xi^\alpha p_1(x,\xi) \right) \left(\partial_x^\alpha p_2(x,\xi) \right).$$

3: Hadamard Parametrix construction.

Theorem: Let D be an elliptic (pseudo-) differential operator of order m. Then there exists a pseudo-differential operator Q of order $-m$, such that $I-QD$, $I-DQ$ are of order $-\infty$ (smoothing).

Proof. Since $\sigma_D(x,\xi)$ is invertible for large ξ, ($\Leftarrow$ D is elliptic).

let $q(x,\xi) = \phi(x,\xi)\,\sigma_D(x,\xi)^{-1}$, where $\phi(x,\xi)$ is the bump function, vanishing for small ξ.

Since $\sigma_D(x,\xi)$ is of order m, <u>by calculation</u>, $\sigma_D(x,\xi)^{-1}$ is of order $-m$.

$\Rightarrow$ $q(x,\xi)$ is a symbol of order $-m$.

Let Q_1 is a pseudo-differential operator with symbol $q(x,\xi)$, so Q_1D is a pseudo-differential operator of order 0.

But from the asymptotic expansion in the previous corollary, the zero order term of Q_1D is I. So define $S = I - Q_1D$.

$\Rightarrow$ S is a pseudo-differential operator of order -1.

Let $X \sim \sum_{j=0}^{\infty} S^j$, so X has order 0.

$\Rightarrow$ $XQ_1D \sim \sum_{j=0}^{\infty} S^j(I-S) = I +$ order$(-\infty)$ terms.

$\Rightarrow$ $XQ_1D \sim I$.

Define $Q_L = XQ_1$, so $Q_LD \sim I$, and Q_L has order $-m$.

Similarly, define Q_R, s.t $DQ_R \sim I$.

$\Rightarrow$ $Q_L \sim Q_LDQ_R \sim Q_R$

So take Q to be either Q_L or Q_R.

· Corollary. (elliptic regularity).

If D is elliptic, then $\operatorname{sing\,supp}(Du) = \operatorname{sing\,supp}(u)$.

So if $Du = v$, $v \in C^\infty \Rightarrow u \in C^\infty$.

Note: Not true for some non-elliptic equations, e.g. wave equation,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \quad \text{has solution} \quad u = f(x+y) \text{ for any } f \in C^1.$$

Proof: Suppose $Du = v \in C^\infty$.

$$u = QDu + (I - QD)u$$

$$= Qv + \underbrace{(I - QD)u}_{\text{smoothing}}$$

From pseudo local property, $\operatorname{sing\,supp}(Qv) \subset \operatorname{sing\,supp} v = \phi$.

$\Rightarrow Qv \in C^\infty$.

$\Rightarrow u \in C^\infty$.

$\Rightarrow \operatorname{sing\,supp}(u) \subseteq \operatorname{sing\,supp}(Du)$

$\Rightarrow \operatorname{sing\,supp}(Du) = \operatorname{sing\,supp}(u)$.