02/10/14

Elliptic Operators and Topology: Lecture 12

Let M be a compact (oriented) Riemannian manifold with metric g, and $f: M \circlearrowleft$ an <u>isometry</u>. Suppose M is even dimensional, so $\dim(M) = m = 2n$, and f has only simple (i.e. isolated) fixed points. The metric g gives rise to a volume form $\mathrm{vol} \in \Omega^m(M)$ and so we have the Hodge isomorphism $*: \Lambda^q T^*M \xrightarrow{\sim} \Lambda^{m-q} T^*M$, which satisfies $\alpha \wedge * \beta = g(\alpha,\beta)\,\mathrm{vol}$. M is even dimensional, so $*^2 = (-1)^q$ on q-forms, and $d^* = -*d*$ is the formal adjoint of d w.r.t. the ~~glob~~ global inner product

$$(\alpha,\beta) = \int_M \alpha \wedge * \beta .$$

Define a map $\varepsilon: \Lambda^q T^*M \longrightarrow \Lambda^{m-q} T^*M$ $\forall q$, and

$$\alpha \longmapsto (-1)^{\frac{(q-1)q}{2}} * \alpha$$

Note that

$$\varepsilon^2(\alpha) = (-1)^{\frac{(q-1)q}{2}} (-1)^{\frac{(m-q-1)(m-q)}{2}} (-1)^q \alpha$$

$$= (-1)^{\frac12 q^2 - \frac12 q + \frac12 (m-q)^2 - \frac12 (m-q) + q} \alpha$$

$$= (-1)^{q^2 + q + 2n^2 - 2nq - n} \alpha$$

$$= (-1)^n \alpha,$$

So $\varepsilon^2 = (-1)^n$ is independent of q.

Now, ε and the ~~[illegible]~~ de Rham operator $D = d + d^*$ <u>anticommute</u>, as

$$\varepsilon D(\alpha) = \varepsilon(d\alpha) - \varepsilon(*d*\alpha) = (-1)^{\frac{q(q+1)}{2}} * d\alpha \,\cancel{-}\, (-1)^{\frac{q(q-1)}{2}} d*\alpha$$

and

$$D\varepsilon(\alpha) = (-1)^{\frac{q(q-1)}{2}} d*\alpha - (-1)^{\frac{q(q+1)}{2}} *d\alpha = -\varepsilon D\alpha .$$

So we may define the <u>signature operator</u>, as

$$D : \Omega^+(M) \to \Omega^-(M),$$

for n even (i.e. $4 | m$) and $\Omega^\pm(M)$ = the $\pm$ eigenspace of ε. If n is odd we get a similar splitting for $i\varepsilon$, so just consider the operator $i^n\varepsilon$. So D is of the form

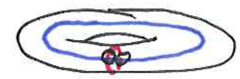
$$D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}$$

with respect to this splitting.

<u>Theorem</u> The index of D (for n even) is the <u>signature</u> of M (i.e. the Hirzebruch signature), and is 0 for n odd.

The <u>intersection form</u> of M is a ~~bilinear form~~ bilinear form

$$Q : H_n(M;\mathbb{Q}) \otimes H_n(M;\mathbb{Q}) \to \mathbb{Q},$$



Skew-symmetric for n odd and symmetric for n even. The signature of M is the signature of this intersection form. Giving such a bilinear form is equivalent, by Poincaré duality, to giving a bilinear form $H^n(M;\mathbb{R}) \otimes H^n(M;\mathbb{R}) \to H^m(M;\mathbb{R})$ and then integrating over M. In our case, this is the cup product, and $Q(\alpha,\beta) \triangleq \int_M \alpha \wedge \beta$ is the required bilinear map on ~~coh~~ cohomology: $Q(\alpha,\beta) = (-1)^{q(m-q)} Q(\beta,\alpha) = (-1)^{n^2} Q(\alpha,\beta)$ for $\alpha,\beta \in \Omega^n(M)$, and $Q(\alpha + d\gamma, \beta) = Q(\alpha,\beta)$, by Stokes' theorem and $d\beta = 0$, for $[\alpha], [\beta] \in H^n(M;\mathbb{R})$. So Q gives a map on cohomology, as claimed.

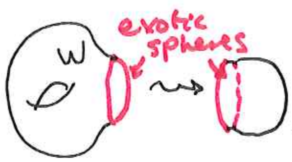
Digression

<u>Theorem</u> (Thom) If $M = \partial W$, then $\mathrm{sign}(M) = 0$.

This implies that ~~sign is a~~ $M \mapsto \mathrm{sign}(M)$ is a function on the <u>oriented cobordism ring</u> (where $M \sim N$ if $M \cup N = \partial W$, for some W). Hirzebruch proved that, e.g. in dim 8, that

$$\mathrm{sign}(M) = \frac{1}{45} p_1^2 - \frac{7}{45} p_2 .$$

Milnor's proof of exotic spheres:



<u>Proof of theorem</u> We will use Hodge theory. Let $\mathcal{H}$ = space of harmonic functions. Then let index = trace$(i^n \varepsilon|_{\mathcal{H}})$. What is this?

We can restrict our attention to the middle dimension of M in cohomology. The intersection form $Q : H^n(M;\mathbb{R}) \otimes H^n(M;\mathbb{R}) \longrightarrow \mathbb{R}$ will satisfy

$$Q(\alpha, i^n \varepsilon(\alpha)) = \int_M \alpha \wedge (i^n \varepsilon(\alpha)) = i^n \int_M \alpha \wedge * \alpha = i^n \|\alpha\|_{L^2} .$$

So $Q(\alpha, i^n \varepsilon(\alpha)) = \pm \|\alpha\|^2_{L^2}$, and the sign depends on the parity of n (mod 4), i.e.

$$Q(\alpha, i(\alpha)) = \begin{cases} \|\alpha\|_{L^2} & n \equiv 0 \bmod 4 \\ -\|\alpha\|_{L^2} & n \equiv 2 \bmod 4 \end{cases}$$

So the splitting of $\mathcal{H}^n = \{\text{harmonic } n\text{-forms}\}$ into $\mathcal{H}^n_+$ and $\mathcal{H}^n_-$, the $+1$ and -1 eigenspaces of $i^n\varepsilon$, give that

$$\mathrm{sign}(Q) = \dim(\mathcal{H}^n_+) - \dim(\mathcal{H}^n_-) = \cancel{\dim\ker(D^+) - \dim\ker(D^-)}$$
$$\dim\ker(D - I) - \dim\ker(D + I)$$
$$= 0$$

So $\operatorname{index}(D) = 0$. QED

What does the Lefschetz theorem say when applied to the signature operator?

Recall that $f: M \circlearrowleft$ is an isometry, and $L(f,M) = \sum_{\{p:\, f(p)=p\}} \nu(p)$.

In our case, the complex is $D: \Omega^{even}(M) \to \Omega^{odd}(M)$ and $L(f,M)$ is the G-signature of f (G a Lie group), a topological invariant. If $p \in \operatorname{Fix}(f)$, then $T_pf: T_pM \circlearrowleft$ is an orientation preserving isometry, so $V = T_pM$ decomposes as a direct sum $\underset{\theta_1}{\circlearrowleft V_1} \oplus \dots \oplus \underset{\theta_n}{\circlearrowleft V_n}$ s.t. $T_pf|_{V_k}$ is a rotation by an angle θ_k, and $V_k \perp V_l$ (by the Spectral theorem).

<u>Theorem</u> $\nu(p) = i^n \prod_{k=1}^{n} \cot(\theta_k/2)$

<u>Proof</u> Next time. $\square$

<u>Application</u>: $f: M \circlearrowleft$ an isometry of order $q = p^l$ (i.e. $f^q = id_M$), p an odd prime. Then ~~if has at~~ f cannot have only one fixed point:

$$\cot(\tfrac{1}{2}\theta) = i\left(\frac{e^{i\theta/2} + e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}\right) = i\left(\frac{e^{i\theta}+1}{e^{i\theta}-1}\right)$$

$$\Rightarrow L(f,M) = \nu(p) = (-1)^n \prod_{k=1}^{n}\left(\frac{1+e^{i\theta_k}}{e^{i\theta_k}-1}\right)$$

$$\Rightarrow L(f,M)\prod_{k=1}^{n}\left(1-e^{i\theta_k}\right) = \prod_{k=1}^{n}\left(1+e^{i\theta_k}\right).$$ Let ψ be a primitive q^{th} root of unity and apply Frobenius l times $\Rightarrow 0 \equiv 2^n \pmod p$. So $p \neq 2$ gives a contradiction.