Elliptic Operators and Topology

Lecture 13, 2014. 10.7

- Signature operator: $D = d + d^* : \Lambda^+ \longrightarrow \Lambda^-$.

Lefschetz fixed point theorem for signature operator:

$$L(f, D) = \sum_{f(p)=p} \nu(p).$$

- Now choose $T_pf : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$, and a system of angles for the isometry T_pf, then $\nu(p) = i^{-n} \prod_{k=1}^{n} \cot\left(\frac{\theta_k}{2}\right)$.

pf: Recall that we are dealing with an elliptic complex of length 2, defined by the ± 1 eigenspaces of the involution $i^n \varepsilon = \eta$. The local contribution is

$$\nu(p) = \frac{\mathrm{Trace}(\eta \Lambda^* T_p^* f)}{|\det(I - T_pf)|}$$

$$= \frac{1}{|\det(I - T_pf)|} \left(\mathrm{Trace}(\Lambda^* T_p^* f \text{ on } \Lambda^+) - \mathrm{Trace}(\Lambda^* T_p^* f \text{ on } \Lambda^-) \right).$$

We must compute $\nu(U)$, where U is the isometry of $2n$-dimensional Euclidean space.

- Lemma: $\nu(U_1 \oplus U_2) = \nu(U_1)\nu(U_2)$.

So let's work out the contribution from a single 2-plane spanned by e and e', on which T_pf acts as $\begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix}$.

$$\Rightarrow \quad * \begin{cases} 1 \longmapsto e\wedge e' \\ e \longmapsto e' \\ e' \longmapsto -e \\ e\wedge e' \longmapsto 1 \end{cases} \quad \Rightarrow \tau \begin{cases} 1 \longmapsto e\wedge e' \\ e \longmapsto e' \\ e' \longmapsto -e \\ e\wedge e' \longmapsto -1 \end{cases}.$$

So Λ^+ is spanned by $\{1+ie\wedge e',\ e+ie'\}$

Λ^- is spanned by $\{1-ie\wedge e',\ e-ie'\}$.

Under this basis, the action of $\Lambda^* T_p^* f$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ on } \Lambda^+, \quad \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} \text{ on } \Lambda^-.$$

Thus $\mathrm{Trace}(\eta \Lambda^* T_p^* f) = e^{-i\theta} - e^{i\theta} = -2i\sin\theta$,

$\det(I - T_p f) = (1-e^{-i\theta})(1-e^{i\theta}) = 2(1-\cos\theta)$.

Then

$$\nu(p) = \frac{\mathrm{Trace}(\eta\Lambda^* T_p^* f)}{|\det(I - T_p f)|} = \frac{-2i\sin\theta}{2-2\cos\theta} = -i\cot\left(\frac{\theta}{2}\right).$$

- Clifford Algebra, Dirac Operators.

Let V be a real vector space with a symmetric bilinear form $\langle\ ,\ \rangle$.

- Def. The Clifford algebra $Cl(V)$ is the algebra generated linearly by V, subject to the relation

$$vw + wv = -2\langle v, w\rangle 1.$$

In other words, it is the quotient of the free tensor algebra $T(V)$ by the ideal generated by the relation.

- Examples / Facts.

1) If $\langle\, ,\, \rangle = 0$, we get $\Lambda^* V$.

2) If $\dim V = n$, then $\dim Cl(V) = 2^n$.

3) $Cl(V)$ is a $(\mathbb{Z}/2)$-graded algebra; $Cl(V) = Cl^{even} \oplus Cl^{odd}$.

4) $Cl(\mathbb{R}) = \mathbb{C}$, $Cl(\mathbb{R}^2) = \mathbb{H}$.

$Cl(\mathbb{R}^3) = \mathbb{H} \oplus \mathbb{H}$.

5) $Cl(\mathbb{R}^{2k}) \otimes \mathbb{C} \cong M_{2^k}(\mathbb{C})$.

- The grading of $Cl(2k)$ (Clifford algebra of $\mathbb{R}^{2k}$) is internal, i.e. even/odd elements are those which commute/anticommute with an involution $\eta = i^k e_1 e_2 \cdots e_{2k}$, where $\{e_1, \cdots, e_{2k}\}$ is an orthonormal basis of $\mathbb{R}^{2k}$.

- Observe that $\eta^2 = 1$, and $e_i \eta = -\eta e_i$.

- Def. A Clifford module W is a (left) module over the Clifford algebra. $Cl(V)$, i.e. for each $v \in V$, there is a linear map $c(v): W \to W$, such that $c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle I$.

Example: Euclidean V, $W = \Lambda^* V$.

$c(v)$ = exterior multiplication by v + interior multiplication by v.

- Dirac Operator.

M Riemannian manifold (oriented).

$$\begin{tikzcd} TM \arrow[d] \arrow[r, squiggly] & \mathrm{Cliff}(TM) \arrow[d] \arrow[r, squiggly] & W \arrow[d] \\ M & M & M \end{tikzcd}$$

W ← Clifford bundle.

- Def: A (generalized) Dirac operator for W is a 1st order differential operator, whose symbol is Clifford multiplication $c(\xi)$.

Note: This operator is elliptic, since $c(\xi)^2 = -|\xi|^2 I$.

Example: $D = d + d^*$.

- More precise version.

Assume W carries a connection ∇ and a hermitian metric, ~~with compatibility~~ such that.

— The Clifford action $c(\xi)$ is skew-adjoint, i.e. $c(\xi)^* = -c(\xi)$.

— The connections are compatible in the sense that

$$\nabla_X(Ys) = (\nabla_X Y)s + Y(\nabla_X s),$$

for each vector fields X, Y and section s, and $\nabla_X Y$ is the Levi-Civita connection.

Then we can define the Dirac operator by

$$D: C^\infty(W) \xrightarrow{\nabla} C^\infty(T^* \otimes W) \xrightarrow{g} C^\infty(T \otimes W) \xrightarrow{c} C^\infty(W),$$

i.e. locally $Ds = \sum_i e_i \nabla_i s$, where $\{e_i\}$ is an orthonormal frame.

· Lemma: This D is formally self-adjoint.

Pf. By local calculation.

$$(Dw_1, w_2) - (w_1, Dw_2) = d^* w,$$

where $w(x) = (c(x)w_1, w_2)$ is 1-form.

Since
$$\int_M d^* w \, vol = \pm \int_M (*1) \wedge (*d*) w$$
$$= \pm \langle vol, d*w \rangle$$
$$= \pm \int_M d*w = 0 \quad \text{(Stokes Theorem)},$$

then $\langle Dw_1, w_2 \rangle = \langle w_1, Dw_2 \rangle$.