Elliptic Operators and Topology: Lecture 14

Today: Spin

$$\begin{tikzcd} \mathrm{Cliff}(M)\otimes W \arrow[r] \arrow[dr] \arrow[d] & W \arrow[dl] \\ M & \end{tikzcd}$$

a Clifford bundle over M.

Recall: $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^{2n}) = M_{2^n}(\mathbb{C}) = \Delta \otimes \Delta^*$

$\Delta \cong \mathbb{C}^{2^n}$.

Principal Bundles

G - a Lie group

M - a manifold

A <u>G-principal bundle</u> over M, $P \to M$, is a locally trivial bundle with fiber G (considered as a right G-space) and transition functions given by left multiplication by sections $U \cap V \to G$.

<u>Example</u> Let $E \to M$ be a real vector bundle. A <u>frame</u> for E_x is an isomorphism from E_x to $\mathbb{R}^k$ (i.e. a choice of basis). The collection of frames of E is a $GL(k;\mathbb{R})$ principal bundle (<u>frame bundle</u>). The underlying vector bundle can be recovered from its frame bundle: consider $P \to E$ the frame bundle, and ~~consider~~ denote $P \times_G \mathbb{R}^k = P \times \mathbb{R}^k/\sim$ where $(pg, v) \sim (p, gv)$ $\forall p \in P, v \in \mathbb{R}^k$ and $g \in GL(k;\mathbb{R})$. Then $P \times_G \mathbb{R}^k \cong E$, as both locally are isomorphic to $M \times \mathbb{R}^k$ and we can take the same transition functions. If we just had a representation $\rho: G \to GL(k;\mathbb{R})$, we could ~~stuff~~ still form a vector bundle by $(pg, v) \sim (p, \rho(g)v)$.

Using a metric, we can just consider orthonormal, ordered bases. This reduces the structure group from $GL(k;\mathbb{R})$ to $SO(k)$, and ~~the~~ so

$$\begin{tikzcd} SO(k) \arrow[r] \arrow[d] & P_{SO} \arrow[r] \arrow[d] & M \arrow[d, equal] \\ GL(k;\mathbb{R}) \arrow[r] & P_{GL} \arrow[r] & M \end{tikzcd}$$

commutes. Replacing $GL(k;\mathbb{R})$ by $O(k)$ gives

$$\begin{tikzcd} SO(k) \arrow[r] \arrow[d] & P_{SO} \arrow[r] \arrow[d] & M \arrow[d, equal] \\ O(k) \arrow[r] & P_{O} \arrow[r] & M. \end{tikzcd}$$

Now, ~~the first vertical map~~ $\pi_1(SO(k)) \cong \mathbb{Z}/2$, so the universal cover ~~of~~ of $SO(k)$ exists and inherits a group structure from $SO(k)$. Denote this ~~universal~~ universal cover by $Spin(k)$, so $Spin(k)/\mathbb{Z}/2 \cong SO(k)$.

We have a representation $\rho : SO(k) \to O(Cliff(k))$ given by

$$\rho(A)\cdot(v_1 \cdot \ldots \cdot v_s) = (Av_1)\cdot\ldots\cdot(Av_s)$$

and for $\{e_{i_1}\cdot\ldots\cdot e_{i_r} \mid 0 \le r \le k, 1 \le i_1 < \ldots < i_r \le k, \{e_i\}_{i=1}^k \text{ an orthonormal basis of } \mathbb{R}^k\}$ an orthonormal basis to $Cliff(k)$,

$$(\rho(A)e_I, \rho(A)e_J) = a^{\alpha_1}_{i_1}\cdots a^{\alpha_r}_{i_r}\, a^{j_1}_{\beta_1}\cdots a^{j_s}_{\beta_s}(e_\alpha, e_\beta) = \delta_{IJ} = (e_I, e_J)$$

so $\rho(A)$ is orthogonal w.r.t. the inner product inherited from $\mathbb{R}^k$.

In the $k=3$ case,

$$\begin{tikzcd} S^3 = Spin(3) = \text{unit quaternions} = \{u \in \mathbb{H} \mid u\bar{u} = \bar{u}u = 1\} \arrow[d] \\ SO(3) \end{tikzcd}$$

and this acts by $x \mapsto ux\bar{u} = uxu^{-1}$, for $x \in Im(\mathbb{H}) \cong \mathbb{R}^3$.

In general, define $Spin(k)$ as follows: for $x \in Cl(k)$, $x = e_1 \cdots e_r$, set $\bar{x} = (-1)^r e_r e_{r-1} \cdots e_1$, and extend linearly to an involution of $Cl(k)$. $Spin(k)$ is the group of $u \in Cl(k)$ s.t. (group under multiplication)

(a) u is even

(b) $u\bar{u} = \bar{u}u = 1$

(c) $x \mapsto ux\bar{u}$ maps $\mathbb{R}^k$ to itself.

(c) then gives a rep. $Spin(k) \to SO(k)$.

<u>Example</u> $u = \cos(\psi) + \sin(\psi) e e' \Rightarrow \bar{u} = \cos(\psi) + \sin(\psi) e'e$
$= \cos(\psi) - \sin(\psi) e e'$,

$$\begin{aligned}
\text{so } \; u e \bar{u} &= (\cos(\psi) + \sin(\psi) ee')\, e\, (\cos(\psi) + \sin(\psi)\, e'e) \\
&= (\cos^2(\psi) - \sin^2(\psi))\, e + 2\sin(\psi)\cos(\psi)\, e' \\
&= \cos(2\psi)\, e + \sin(2\psi)\, e',
\end{aligned}$$

$$\begin{aligned}
\text{and } \; u e' \bar{u} &= \cancel{\cos(\psi) - \sin(\psi)} \; (\cos(\psi) + \sin(\psi) ee')\, e' (\cos(\psi) + \sin(\psi) e'e) \\
&= -\sin(2\psi)\, e + \cos(2\psi)\, e'
\end{aligned}$$

$\Rightarrow u \mapsto \begin{pmatrix} \cos(2\psi) & -\sin(2\psi) \\ \sin(2\psi) & \cos(2\psi) \end{pmatrix} \in SO(2)$ is a rotation by 2ψ in $\mathbb{R}^2$.

$Spin(2n)$ has a representation $\Delta = \overset{\text{even}}{\Delta_+} \oplus \overset{\text{odd}}{\Delta_-}$
irred. rep. of Spin (half spin)

<u>Def:</u> a <u>spin structure</u> on an oriented vector bundle E (w/a metric) is a "reduction" of the structure group from $SO(k)$ to $Spin(k)$,

i.e.

$$\begin{tikzcd}
Spin(k) \arrow[r, "\text{double}", "\text{cover}"'] \arrow[d] & SO(k) \arrow[d] \\
P_{Spin}(E) \arrow[r, dashed, "\exists"] \arrow[d] & P_{SO}(E) \arrow[d] \\
M \arrow[r, equal] & M
\end{tikzcd}$$

A <u>spin structure</u> on a <u>Riemannian manifold</u> is a structure on TM.

The Atiyah-Singer-Dirac Operator

Let M^{2n} be a spin manifold, with spin ~~bundle~~ frame bundle P_{Spin}. Define $S^{\pm} = P_{Spin} \times_\rho \Delta_{\pm}$, where ρ is the is the spin rep. on Δ, and let $S = S^+ \oplus S^- = P_{Spin} \times_\rho \Delta$. Then S is a bundle of Clifford modules. The corresponding Dirac operator

$$D : C^\infty(S^+) \to C^\infty(S^-)$$

defines an elliptic complex of length 2. This construction is unique, once P_{Spin} is fixed, and to distinguish it from other Dirac operators it is often referred to as the Atiyah-Singer-Dirac operator.

Fixed Point Theorem

M a spin manifold (even dim.), $f: M \circlearrowleft$ an isometry, $\hat{f}: P_{Spin} \circlearrowleft$ a spin lifting coming from the action of the isometry Tf on the orthonormal frame bundle, i.e. $Tf: SO(TM) \circlearrowleft \to$ ~~lift~~ lift this to P_{Spin}.

The contribution from an isolated fixed point p in the LFPT is

$$\pm i^n 2^{-n} \prod_{k=1}^{n} \operatorname{cosec}(\theta_k/2)$$

where the θ_k are the angles that the isometry T_pf rotates each line by.

Outline of the proof

$$\nu(p) = \frac{1}{|\det(I - T_pf)|}\left(\text{Trace on } \Delta^+ - \text{Trace on } \Delta^-\right)$$

Consider the basis elements $\{e_1, e_1', \ldots, e_n, e_n'\} = \{e_I \mid 0 \le |I| \le 2n\}$.

$1 \mapsto \mathrm{Id}$, but $e_i \mapsto$ something skew-symmetric

$\Rightarrow \mathrm{Trace}_{\mathrm{Cliff}}(1) = 2^n$, but $\mathrm{Trace}_{\mathrm{Cliff}}(\cancel{\;\;}) = 0$ on the other basis elements.