16/10/14

Elliptic Operators and Topology: Lecture 15

Fixed point contribution for the Dirac operator.

Recall: M^{2n}, $\nu(p) = \pm\, i^n 2^{-n} \prod_{k=1}^{n} \operatorname{cosec}(\theta_k/2)$

for $T_pf : T_pM \circlearrowleft$ an isometry, consisting of rotations by angles $\theta_1, \ldots, \theta_n$. Where does this formula come from?

$\nu(p) = \dfrac{1}{|\det(I - T_pf)|} \operatorname{Trace}(\eta \widetilde{T_pf})$, where $\eta = i^n e_1 e_1' \cdots e_n e_n'$ is the usual grading operator and $\widetilde{T_pf}$ is a lifting of T_pf to $(P_{spin})_p \circlearrowleft$.

$$T_pf = \begin{pmatrix} R(\theta_1) & & \\ & \ddots & \\ & & R(\theta_n) \end{pmatrix} \quad \text{for} \quad R(\theta_k) = \begin{pmatrix} \cos(\theta_k) & -\sin(\theta_k) \\ \sin(\theta_k) & \cos\theta_k \end{pmatrix}$$

$$\Rightarrow \det(I - T_pf) = \prod_{k=1}^{n} \det(I - R(\theta_k)) = \prod_{k=1}^{n} \left[(1 - \cos\theta_k)^2 + \sin^2\theta_k\right]$$

$$= \prod_{k=1}^{n} 4\sin^2(\theta_k/2) = 2^{2n} \prod_{k=1}^{n} \sin^2(\theta_k/2).$$

What about the trace?

We are interested in understanding the trace of ηg for $g \in Spin(2n)$ acting on the spin representation.

Now, $Cliff(V)$ decomposes as a left $Cliff(V)$-module as

$$\Delta \otimes \Delta^* = \bigoplus^{2^n} \Delta.$$

Think of an element of $Cliff(V)$ acting on itself (by left regular rep^n) and we compute the trace as follows: an orthonormal basis of $Cliff(V)$ consisting of $1, e_1, e_1', \ldots, e_1 \cdots e_n'$, i.e. e_I for

$I \subseteq \{1, 1', \dots, n, n'\}$ & $e_\emptyset = 1$, so that

$\mathrm{Tr}(1) = \mathrm{Tr}(I) = 2^n$

and e_i are skew-symmetric, as

$(e_i \cdot e_I, e_J) = (-1)^r$ iff $\{i\} \cup I = J$ & $i = j_r$, $i_s = j_s \ \forall s < r$,

and $e_i \cdot e_J = (-1)^r e_I$ & $(e_i e_I, e_J) = 0 = (e_I, e_i e_J)$ otherwise,

so $\mathrm{Tr}(e_i) = 0$ & e_I is skew-symmetric $\Rightarrow \mathrm{Tr}(e_I) = 0$, for $I \neq \emptyset$.

So $\mathrm{Tr}(\eta g) = 2^n \cdot (\text{coeff. of } 1 \text{ in } \eta g)$
$= 2^n \cdot (\pm \text{coeff of } \eta \text{ in } g)$.

A spin lifting of $T_p f$ is $\prod_{k=1}^{n} \left(\cos(\theta_k/2) + e_k e_{k'} \sin(\theta_k/2)\right)$ (coming from the double cover $\left(\cos(\theta_k/2) + e e' \sin(\theta_k/2)\right)(\cdot)\overline{\left(\cos(\theta_k/2) + e e' \sin(\theta_k/2)\right)}$ of $SO(2n)$ by $Spin(2n)$) so

$$\mathrm{Tr}(\eta \widetilde{T_p f}) = \frac{i^n 2^n \prod_{k=1}^{n} \sin(\theta_k/2)}{4^n \prod_{k=1}^{n} \sin^2(\theta_k/2)} = i^n 2^{-n} \prod_{k=1}^{n} \csc(\theta_k/2).$$

Heat Operator

M- compact

S- spin bundle

D- Dirac operator (generalized)

$\to$ Consider the heat operator e^{-tD^2}

What does this mean?

Answer: the space $L^2(S)$ (S has a hermitian metric) can be written as a direct sum of eigenspaces (orthogonal).

E_λ, with $\lambda_i \to \infty$ & $\lambda_i \geq 0 \ \forall i \geq 1$, of the operator D^2.

Moreover, each E_λ is finite dimensional and consists of smooth sections
$\Rightarrow$ $s \in L^2(S)$ is smooth iff its coefficients w.r.t. this (orthonormal) basis are rapidly decreasing.

<u>Example</u> Think about $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $D = i\frac{d}{dx} \Rightarrow D^2 = -\frac{d^2}{dx^2}$.

Then $\{\lambda_n\}_{n\geq 0}$ is $0, 1, 1, 4, 4, \dots$
$$1 \quad e^{i\theta} \quad e^{-i\theta} \quad e^{2i\theta} \quad e^{-2i\theta}$$

as $D^2 f = \lambda f \Rightarrow f'' = -\lambda f$
$$\Rightarrow (f'^2 + \lambda f^2)' = 0$$
So $f' = \sqrt{C - \lambda f^2} \Rightarrow f = A e^{i\sqrt{\lambda}\theta} + B e^{-i\sqrt{\lambda}\theta}$

s.t. $\sqrt{\lambda} \in \mathbb{Z} \Rightarrow \lambda \in \{0, 1, 4, 9, 16, \dots\}$.

<u>Proof</u> Let $Q = (D^2 + I)^{-1}$, which is compact and self-adjoint on L^2. Use the spectral theorem, then reverse engineer this to get D^2.

For a function f, define
$$f(D^2) = \sum_j f(\lambda_j) P_j$$
where P_j is the projection onto E_{λ_j}. It follows that e^{-tD^2} is a smoothing operator, with smoothing kernel $k_t(x,y) \in S_x \otimes S_y^*$, the <u>heat kernel</u>.

Recall: in Euclidean space,
$$h_t(x,y) = \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}.$$

On a manifold M, let $k_t(x,y) = \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \exp\left(-\frac{d_M(x,y)^2}{4t}\right)$, where
$d_M(x,y)$ = distance between x and y in the metric arising for the metric tensor g.

Fundamental Asymptotic Theorem

There is an asymptotic series of the following sort:

$$k_t(x,y) \sim h_t(x,y)\left(\Theta_0(x,y) + t\,\Theta_1(x,y) + \frac{t^2}{2!}\Theta_2(x,y) + \cdots\right)$$

such that $\Theta_0(x,x) = Id_{S_x}$.

$\text{Index}(D) \sim \text{Trace}(k_t(x,y)) \sim \left(\frac{1}{4\pi t}\right)^{\frac{n}{2}} \sum_{k=0}^{\infty} t^k \int_M \Theta_k(x,x)\,dx$, but ~~determines~~

Index (D) is constant, so this determine the $\int_M \Theta_k(x,x)\,dx$.