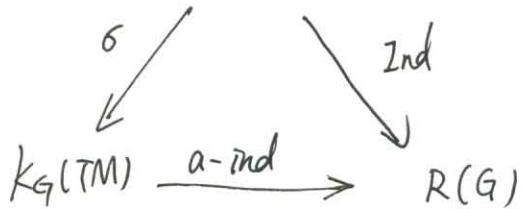


Elliptic operators and Topology

Lecture 18, 2014. 10. 28

Elliptic Operator



Problem: What is $\alpha\text{-ind}$? (in terms of more conventional topology).

Solution: ① construct $t\text{-ind} : K_G(TM) \rightarrow R(G)$;

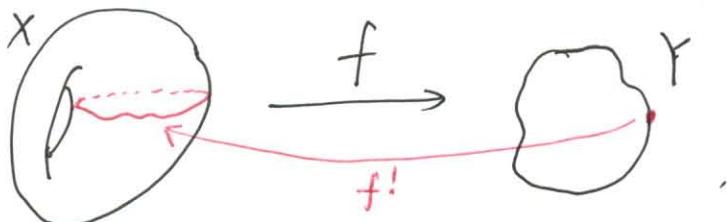
② characterize $t\text{-ind}$ axiomatically.

③ show $\alpha\text{-ind}$ satisfies the axioms.

Shrieking ("wrong way" maps).

Usually $f : X \rightarrow Y$ induces $f_* : H_*(X) \rightarrow H_*(Y)$. But sometimes we can also have a map $f^! : H_*(Y) \rightarrow H_{*-k}(X)$.

For example, if



We can construct $f^! : H_0(Y) \rightarrow H_1(X)$.

$t\text{-ind}$ is constructed from "shriek" maps in K -theory.

- k is a contravariant functor for proper maps, but there are also cases when we can define a covariant ("wrong way") functoriality for certain maps.

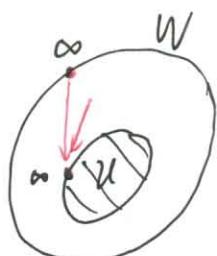
- Examples.

- 1) open inclusions.

Let W be compact or locally compact, and U is an open subset in W . Then there is a map of 1-point compactifications

$$W^+ \xrightarrow{\cong} U^+,$$

which induces a "wrong way" map $k(U^+) \rightarrow k(W^+)$.



So we get a map $i! : k(U) \rightarrow k(W)$, where i is the inclusion $i : U \rightarrow W$.

- 2) X compact or locally compact, V complex vector bundle over X .

The Thom map $\varphi_V : k(X) \rightarrow k(V)$ can be regarded as $j!$, where $j : X \rightarrow V$ is the zero section.

- Remark: Both of these are functorial constructions. If $U \subseteq W \subseteq X$, then

the diagram
$$\begin{array}{ccc} k(U) & \longrightarrow & k(W) \\ \downarrow & \swarrow & \text{commutes.} \\ k(X) & & \end{array}$$

- Combining these.

Suppose $j : M \rightarrow M'$ is an embedding of manifolds.

Tubular neighborhood Theorem: There is a neighborhood of M in M' , which is identified with the total space of the normal bundle,

$$N = (TM')|_M / \pi_M^* TM,$$

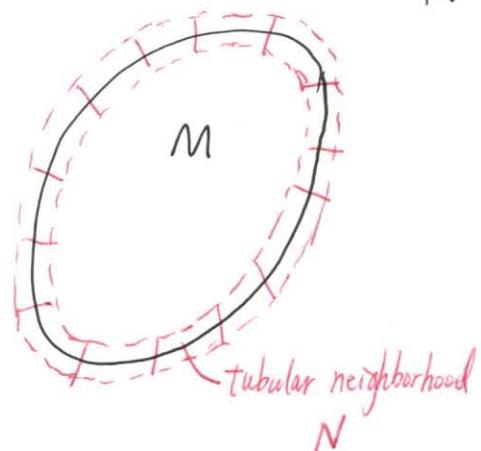
\Downarrow
 $j^*(TM')$

in such a way that M is identified with the zero section.

If N is provided with a complex structure, then we can define.

$$k(M) \xrightarrow{\text{Thom}} k(N)$$

$$\begin{array}{ccc} & & \downarrow \text{open inclusion} \\ j! & \searrow & \downarrow \\ & & k(M') \end{array}$$



Now consider map $TM \rightarrow TM'$, then the normal bundle of TM in TM' is $\pi^*(N \oplus N)$, where π is the projection $TM \xrightarrow{\pi} M$, and $N \oplus N$ always has a complex structure $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus we can always define a "wrong way" map $j! : k_g(TM) \rightarrow k_g(TM')$. We will use this to define $t\text{-ind}$.

- Proposition: Every compact manifold M (compact G -manifold M) can be embedded in an Euclidean space E (an orthogonal representation space of G).

- Definition of $t\text{-ind}$: $j : M \rightarrow E$ is an embedding, then the

Suppose

topological index is a map

$$K_G(TM) \xrightarrow{j!} K_G(TE) \xleftarrow{\cong} K_G(pt) = R(G)$$

Bott isomorphism

$t\text{-ind}$

- Lemma: This definition of $t\text{-ind}$ doesn't depend on choice of E .

Proof: Suppose we have two embeddings $j: M \rightarrow E$ and $j': M \rightarrow E'$.

Define $k(x) = (j(x), j'(x)): M \rightarrow E \oplus E'$.

To see j and j' define the same $t\text{-ind}$, it is sufficient to show that j and k define the same $t\text{-ind}$.

Now we have a homotopy of embeddings $*: M \rightarrow E \oplus E'$ given by

$$k_s(x) = j(x) \oplus s j'(x) = (j(x), s j'(x)),$$

and $t\text{-ind}$ depends only on the homotopy class. Thus it will be enough to compare j with k_0 . If N is the normal bundle of $j(M)$, then the normal bundle of $k_0(M)$ is $N \oplus E'$. Thus we have a commutative diagram, because of the transitivity of Thom map. Also, a similar result holds, if M is replaced by a point.

$$\begin{array}{ccc} K_G(TM) & & \\ j! \swarrow & & \searrow k_0! = k! \\ K_G(TE) & \xrightarrow{\text{Thom}} & K_G(T(E \oplus E')) \\ & \nwarrow & \nearrow \\ & K_G(pt) = R(G) & \end{array}$$

This commutative diagram proves the lemma. (See section 3).

- Axioms for index functions
- Definition: An index map $\underline{\text{ind}}$ is a collection of G -module homomorphisms $K_G(TM) \rightarrow R(G)$ (for each compact G and each compact G -manifold M), which are natural w.r.t homomorphisms (on the level of G) and G -equivariant diffeomorphisms (on the level of M).
- Axioms
 - (A1) If $M = \text{point}$, then $\text{ind} = \text{id}_{R(G)}$.
 - (A2) ind is natural, w.r.t $j_! : K_G(TM) \rightarrow K_G(TM')$, where $j : M \rightarrow M'$ is an embedding.
- Theorem: If an index function has (A1) and (A2), then it is the topological index. (See section 4).

This is more or less obvious because of the way how $t\text{-ind}$ is defined.

The only issue here is the compact G -manifolds in (A2) versus the embeddings in Euclidean spaces in the definition of $t\text{-ind}$. But the action of G on E can be ~~assumed~~ assumed orthogonal (by rotation), then it extends to an action on the compact G -manifold E^+ , which is just the sphere.

- We will give some B axioms: $\underbrace{(B1), (B2), (B3)}_{\rightsquigarrow} (A2)$.