

Last time: Studied two maps

$$a\text{-ind}: K(TM) \rightarrow \mathbb{Z}$$

$$t\text{-ind}: K(TM) \rightarrow \mathbb{Z}$$

and we wish to show that they are the same.

### Index Function

A function  $\text{ind}: K_G(M) \rightarrow R(G)$

Satisfying a list of axioms; in this case

(A1): if  $M = \text{pt.}$ , then  $\text{ind}: K_G(\text{pt}) \cong R(G) \rightarrow R(G)$  is just the identity.

(Ad)  $\text{ind}$  commutes with the shriek map  $j^!: K_G(TM) \rightarrow K_G(TM')$ ,  
for  $j: M' \hookrightarrow M'$  an embedding.

We will find a list (B1)-(B3) s.t. these imply (A2).

As (Ad) involves something (namely,  $j^!$ ) that is rather complicated, we will substitute it by a different, equivalent list.

### B Axioms

(B1) excision

Let  $U$  be a non-compact  $G$ -manifold which is tame - i.e.  
an open subset of a compact  $G$ -manifold  $M$ .

Then, if  $j: U \hookrightarrow M$  and  $j': U \hookrightarrow M'$  are two different tannings of  $U$  (i.e. embeddings into compact  $G$ -manifolds), then there is a commutative diagram

$$\begin{array}{ccccc} & & K_G(TM) & & \\ & j! & \nearrow & \searrow \text{ind}^M & \\ K_G(TU) & \xrightarrow{\text{ind}^U} & R(G) & & \\ & j! & \nearrow & \searrow \text{ind}^{M'} & \\ & K_G(TM') & & & \end{array}$$

i.e. the map  $\text{ind}^M \circ j!$  does not depend on the pair  $(j, M)$ , and is denoted by  $\text{ind}^U$ .

Example  $\mathbb{R}^n$  can be tamed in numerous ways, e.g. by  $\mathbb{R}^n \hookrightarrow \mathbb{T}^n$ ,  $\mathbb{R}^n \hookrightarrow S^n$ ,  $\mathbb{R}^n \hookrightarrow \mathbb{R}\mathbb{P}^n$ , or any combination such as  $\mathbb{R}^n \hookrightarrow \mathbb{T}^{n-k} \times S^k$ . (We should be careful about the group action, though.)

We will be taking the index of elliptic operators on  $M$  that are "trivial" over  $M \setminus U$ , and then restrict  $\circ$  to elliptic operators on  $U$ .

### (B2) Normalization

Let  $G = O(n)$ ,  $b \in K(\mathbb{T}\mathbb{R}^n)$  be the Bott generator

(so  $b = [0 \rightarrow \Lambda^1 T_{\mathbb{R}} \xrightarrow{\sim} \Lambda^1 T_{\mathbb{R}} \xrightarrow{\sim} \dots \rightarrow \Lambda^n T_{\mathbb{R}} \xrightarrow{\sim} 0]$ ) so  
~~(if)~~  $j!(1) = b$ , for  $1 \in K_G(pt) = R(G)$  &  $j: pt \hookrightarrow \mathbb{R}^n$ .

So  $\text{ind}: K_G(\mathbb{T}\mathbb{R}^n) \rightarrow R(G)$ .

Require:  $\boxed{\text{ind}(b) = 1 \in R(G)}$

(B3) Multiplicative Axiom

Special case:  $W = M \times F$  (i.e. a trivial fiber bundle).

Then  $a \in K_G(TM)$ ,  $b \in K_G(TF) \Rightarrow ab \in K_G(TW)$ , by pulling back both bundles to  $W$  & taking their product.

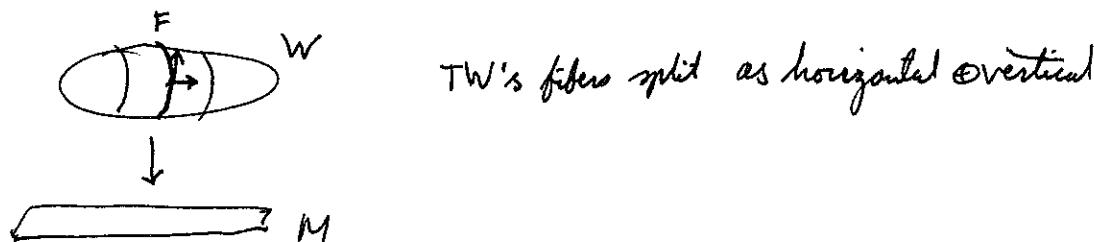
Require:  $\boxed{\text{ind}(ab) = \text{ind}(a) \text{ind}(b)} \in R(G)$

If  $b$  is the Bott generator for  $F = \mathbb{R}^n$ , then this shows that  $\text{ind}(ab) = \text{ind}(a) \Rightarrow \text{ind}$  commutes with the Thom isomorphism  $\Rightarrow$  we have proven what we needed to prove for trivial vector bundles.

General case: suppose  $H$  is another compact group, and that  $P \rightarrow M$  is a principal  $H$ -bundle, and that  $F$  is a  $H$ -manifold. If  $G$  acts on something, we require that the actions of  $G$  and  $H$  commute wherever they are both defined, i.e.

$$g(h \cdot v) = \cancel{gh} h \cdot (g \cdot v).$$

Consider consider  $W = P \times_H F$ , which is by definition an  $F$ -bundle over  $M$ , and so  $F \hookrightarrow W \rightarrow M$ .



Suppose  $a \in K_G(TM)$ ,  $b \in K_{G \times H}(TF)$ .

Now,  $TW = \pi^* TM \oplus T(W/M)$  and we can form

$P_{X_H} TF$  a vector bundle.

$\downarrow$   
 $W$

$T(W/M) = P_{X_H} TF \rightarrow TF$  & hence  $\exists$  a projection  
 $P_X TF \rightarrow TF$ .

So  $K_H(TF) \rightarrow K_H(P_{X_H} TF) \cong K_H(P_X TF) = K(T(W/M))$ ,

hence  $a b \in K_G(TW)$ .

The axiom (B3) then requires that, for  $b \in K_H(TF) \Rightarrow \text{ind}(b) \in R(H)$ ,

that  $\text{ind}(ab) = \boxed{\text{ind}(a) \cdot \mu \text{ind}(b)}$ ,

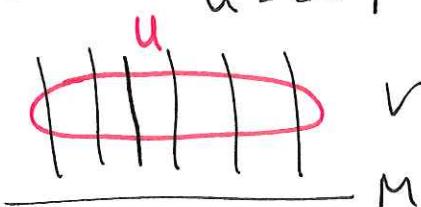
where  $\mu: R(H) \rightarrow K(H)$  is induced by  $P$   
 $E \mapsto P_{X_H} E$

(Here,  $E$  is a representation space of  $H$ , so  $P_{X_H} E$  is well-defined.)

$V$  a real vector bundle. Then  $K(V)$  is a  $K(M)$ -module, so  
 $\downarrow$   
 $M$

$a \in K(V) \Rightarrow a = \{c_U, E_1, E_2\}$  for  $c: E_1 \rightarrow E_2$  &  $c|_U: E_1|_U \xrightarrow{\sim} E_2|_U$ ,

$U \subseteq V$  open.



Proposition (B1)-(B3) imply (A2).

Sketch of proof  $j: M \rightarrow M'$ ,  $K(TM) \rightarrow K(TU)$  for  
 $\bullet \bullet M \subseteq U \subseteq M'$  a tubular neighbourhood, gives  
a commutative diagram

$$j^! : K(TM) \xrightarrow{\text{Thom}} K(TU) \longrightarrow K(TM')$$

↓  
 (B2) (B3)  
 ind      ↓      ↗ (B1)  
 R(G)

(B1)  $\Rightarrow$  right-hand triangle commutes.

(B2) & (B3)  $\Rightarrow$  left-hand triangle commutes.

### Thom Isomorphism

$W$  = a real vector bundle over  $M$

=  $P \times_M \mathbb{R}^n$ , where  $M = \Omega(n)$  and  $P$  = frame bundle

$a \in K(TM)$ ,  $b \in K_{\Omega(n)}(\mathbb{R}^n)$  = Bott generator

$\Rightarrow ab = \psi_W(a)$  (i.e. the ~~the~~ Thom isomorphism), so

ind commutes with the ~~the~~ Thom isomorphism since

~~as~~  $\text{ind}(b) = 1$ , so

~~as~~  $\text{ind}(\psi_W(a)) = \text{ind}(ab) = \text{ind}(a \cdot \mu \circ \text{ind}(b))$

$$\Rightarrow \text{ind}(\psi_W(a)) = \text{ind}(a \cdot \mu(1)) = \text{ind}(a) = \psi_{pt}(\text{ind}(a)),$$

as the zero section of  $pt \rightarrow pt$  is the identity.