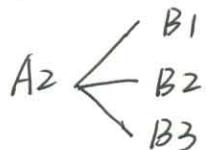


# Elliptic Operators and Topology.

Lecture-22, 2014. 11. 18.

- $a\text{-ind} : K_G(TM) \longrightarrow R(G)$

Axioms : A1



Today, "prove" that  $a\text{-ind}$  satisfies B3, and this will complete the proof of the index theorem.

- Property B3 :

(Simple version) compact  $X, Y$ . If  $a \in k(TX)$ ,  $b \in k(TY)$ ,  
 $\Rightarrow ab \in k(T(X \times Y))$ .

Axiom B3 :  $\text{index}(ab) = \text{index}(a) \text{index}(b)$ .

(Industrial-Strength Version) compact  $X, Y$ , compact group  $H$ , and a principle  $H$ -bundle  $P$  over  $X$ , and an action of  $H$  on  $Y$ .

$$\Rightarrow M = P \times_H Y \longrightarrow X.$$

If  $a \in k(TX)$ ,  $b \in k_H(TY)$ ,

$$\Rightarrow ab \in k(TM).$$

Axiom B3 :  $\text{Index}(ab) = \text{Index}(a \mu_P(\text{index } b))$ , where  
 $\mu_P : R(H) \longrightarrow k(X)$ .

Note: There is also a  $G$ -equivalent version, where  $G$  acts on everything.

- Idea of proof.

$$a \in K(TX) \longrightarrow \text{index}(a) \in \mathbb{Z}.$$

operator  $P$  with symbol  $a$

Step 1: Represent  $a, b$  as symbols of operators  $P, Q$ .

Step 2: Find an operator  $R$  (in terms of  $P$  and  $Q$ ), whose symbol class is  $ab$ .

Step 3: compute index  $R$ .

- Preliminary: understand product in the "symbolic complex" picture of  $K$ -theory.

Answer:  $E_0 \xrightarrow{\alpha_0} E_1 \xrightarrow{\alpha_1} E_2 \xrightarrow{\alpha_2} \dots$  over  ~~$\mathbb{K}TX$~~ .

$F_0 \xrightarrow{\beta_0} F_1 \xrightarrow{\beta_1} F_2 \xrightarrow{\beta_2} \dots$  over  ~~$\mathbb{K}TY$~~ .

Combine:  $E \boxtimes F := \pi_x^* E \otimes \pi_y^* F$ , where  $\pi$  is the projection from  $X \times Y$  to  $X$  or  $Y$ .

$$\begin{array}{ccccccc}
 & & & & & & \\
 & \vdots & & & & & \\
 E_0 \boxtimes F_2 & & & & & & \\
 & \uparrow 1 \boxtimes \beta_1 & & & & & \\
 & & \vdots & & & & \\
 E_0 \boxtimes F_1 & \xrightarrow{\alpha_0 \boxtimes 1} & E_1 \boxtimes F_1 & \longrightarrow & & & \text{make squares} \\
 & \uparrow 1 \boxtimes \beta_0 & & \uparrow -1 \boxtimes \beta_0 & & & \text{anti-commute.} \\
 E_0 \boxtimes F_0 & \xrightarrow{\alpha_0 \boxtimes 1} & E_1 \boxtimes F_0 & \xrightarrow{\alpha_1 \boxtimes 1} & E_2 \boxtimes F_0 & \longrightarrow & \\
 & & & & & &
 \end{array}$$

Define  $G_m = \bigoplus_{i=0}^m E_i \boxtimes F_{m-i}$ . ("symbolic double complex")

$$\gamma_m = \sum_{i=0}^m (\alpha_i \boxtimes 1 + (-1)^i 1 \boxtimes \beta_{m-i}) : G_m \longrightarrow G_{m+1}.$$

• Check:  $\gamma^2 = 0$ .

This complex is exact when  $\alpha$  or  $\beta$  is exact. So this is a symbolic complex over  $TX \times TY$ , which is an element of  $k(T(X \times Y))$ .

- Suppose  $a \in k(TX)$  — symbol of  $p : E_0 \rightarrow E_1$ .  
 $b \in k(TY)$  — symbol of  $Q : F_0 \rightarrow F_1$ .

We will prove product formula when  $P, Q$  are differential operators, by doing with operators  $P, Q$  exactly as we did with their symbols.

$$\begin{array}{ccc} C^\infty(E_0 \boxtimes F_1) & \xrightarrow{P \boxtimes 1} & C^\infty(E_1 \boxtimes F_1) \\ \uparrow 1 \boxtimes Q & & \uparrow -1 \boxtimes Q \\ C^\infty(E_0 \boxtimes F_0) & \xrightarrow{P \boxtimes 1} & C^\infty(E_1 \boxtimes F_0) \end{array}$$

- "Roll up"

$$R : C^\infty((E_0 \boxtimes F_0) \oplus (E_1 \boxtimes F_1)) \longrightarrow C^\infty((E_1 \boxtimes F_0) \oplus (E_0 \boxtimes F_1)).$$

$$R = \begin{pmatrix} P \boxtimes 1 & -1 \boxtimes Q^* \\ 1 \boxtimes Q & P^* \boxtimes 1 \end{pmatrix},$$

where  $P^*, Q^*$  are the adjoint operator of  $P, Q$ .

$$\begin{aligned} R^* R &= \begin{pmatrix} P^* \boxtimes 1 & 1 \boxtimes Q^* \\ -1 \boxtimes Q & P \boxtimes 1 \end{pmatrix} \begin{pmatrix} P \boxtimes 1 & -1 \boxtimes Q^* \\ 1 \boxtimes Q & P^* \boxtimes 1 \end{pmatrix} \\ &= \begin{pmatrix} P^* P \boxtimes 1 + 1 \boxtimes Q^* Q & 0 \\ 0 & P P^* \boxtimes 1 + 1 \boxtimes Q Q^* \end{pmatrix} \end{aligned}$$

Note: we will use the fact:  $\ker R = \ker(R^* R)$ .

$$\begin{aligned}
\text{So } \ker R &= \ker(R^*R) \\
&= \ker(P^*P \boxtimes I + I \boxtimes Q^*Q) \oplus \ker(pp^* \boxtimes I + I \boxtimes QQ^*) \\
&= (\ker(P^*P) \otimes \ker(Q^*Q)) \oplus (\ker(pp^*) \otimes \ker(QQ^*)) \\
&= (\ker P \otimes \ker Q) \oplus (\ker p^* \otimes \ker Q^*).
\end{aligned}$$

Similarly,

$$\ker R^* = (\ker P \otimes \ker Q^*) \oplus (\ker P^* \oplus \ker Q).$$

Then by direct computation,

$$\begin{aligned}
\text{index } R &= \dim \ker R - \dim \ker R^* \\
&= \cancel{\text{index}} (\dim \ker P - \dim \ker P^*) (\dim \ker Q - \dim \ker Q^*) \\
&= \text{Index } P \text{ Index } Q.
\end{aligned}$$

- Problem in PDO case.

If  $D$  is a differential operator, then  $D \boxtimes I$  is also a differential operator.

But if  $P$  is a pseudo-differential operator,  $\nRightarrow P \boxtimes I$  is a pseudo-differential operator.

In fact, this is always false except differential operator case.

- Why not?

- Explanation 1:  $P \boxtimes I$  is not pseudolocal in general.

Consider  $(P \boxtimes I)(\delta_x \boxtimes \delta_y) = P\delta_x \boxtimes \delta_y$ .

- Explanation 2: Let  $p(x, \xi)$  be the symbol of  $P$ .

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C_{\alpha\beta} (1+|\xi|)^{m-|\beta|}.$$

Symbol of  $p \otimes 1$ ? We need

$$|\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma \partial_\eta^\delta p(x, \xi)| \leq C (1+|\xi|+|\eta|)^{m-|\beta|-|\delta|}.$$

Since  $p(x, \xi)$  actually doesn't depend on  $\eta$  at all, this only can happen when  $\partial_\xi^\beta p(x, \xi) = 0$  for  $|\beta| > m$ , i.e.  $p$  is a polynomial. This means  $p$  is a differential operator.

- What substitutes for this is some Sobolev space limiting ideas.