

$X \times Y$ D differential on $X \Rightarrow D \boxtimes 1$ differential on $X \times Y$.P Ψ DO on $X \not\Rightarrow P \boxtimes 1$ Ψ DO on $X \times Y$.How to Fix This?

- 1) All the K-theory, index theory, etc. can be done using limits of Ψ DOs rather than actual Ψ DOs.
- 2) Show that for $m=1$ (what matters is that $m > 0$) the $P \boxtimes 1$ is a limit of Ψ DOs of the required type.

Discussion of Step 1)

Recall the proof (see lecture 10) of the boundedness of Ψ DOs. We proved that, for $p(\alpha, \xi)$ a symbol of order 0, and compactly supported in x , the Ψ DO P corresponding to $p(\alpha, \xi)$ is bounded in $L^2(\mathbb{R}^n)$.

Reminder of how this was provenDefine a constant coefficient operator $q(\xi)$:

$$\widehat{Qu}(\xi) := q(\xi) \widehat{u}(\xi)$$

$$\Rightarrow \|Qu\|_{L^2} = \sup_{\xi \in \mathbb{R}^n} \{ |q(\xi)| \}$$

"Synthesize" P from constant coefficient operators:

$$Pu(x) := \int P_\eta u(x) \cdot e^{ix\cdot\eta} d\eta$$

for P_η = constant coefficient operator w/ symbol

$$\xi \mapsto \hat{p}(z, \xi),$$

$$\text{i.e. } \hat{p}(z, \xi) = \left(\frac{1}{2\pi}\right)^n \int p(x, \xi) e^{-ix\cdot\xi} dx.$$

$$\text{So } \|P_\eta\| = \sup \{ |\hat{p}(z, \xi)| \mid \xi \in T_x^* M \},$$

$$\text{hence } \|P_\eta\| \leq \left(\frac{1}{2\pi}\right)^n \int |p(x, \xi) e^{-iz\cdot x}| dx \leq C_n (1+|z|)^{-N}$$

(integrate by parts), as $|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leq C'(1+|\xi|)^{m-|\beta|}$ and it is compactly supported.

$$\text{So } \|Pu\|_{L^2} \leq \int C (1+|z|)^{-N} \|u\|_{L^2} d\eta = O(\|u\|_{L^2}),$$

$$\text{i.e. } \|Pu\|_{L^2} \leq C'' \|u\|_{L^2}.$$

Notice: we never differentiated anything w.r.t. ξ in this proof.

What this gives us

M compact, \mathcal{P}_0 = algebra of PDOs of order ≤ 0 , \mathcal{P}_1 = ideal in \mathcal{P}_0

(3)

of operators of order ≤ -1 . Hence, we have a short exact sequence

$$0 \rightarrow P_{-1} \xrightarrow{\cap} P_0 \xrightarrow{\cap} S \rightarrow 0$$

\uparrow commutative

$K(L^2)$ $B(L^2)$

compact operators bounded operators

\Rightarrow we can take closures of the above short exact sequence:

$$0 \rightarrow K \rightarrow P_0 \xrightarrow{\text{closure of}} C(Sph(M)) \rightarrow 0$$

compact ops closure of continuous
(as all smoothing P_0 functions on
ops lie in P_1 ,
and there are
clsses in K .) the sphere bundle
of M .

- Notice: (i) operators in P_0 have continuous symbols, which will still define K-theory classes (if elliptic),
- (ii) an elliptic operator in P_0 (i.e. one with an invertible symbol) is Fredholm, \Rightarrow hence ~~P~~ has an index.

Step 2)

Lemma Let P be a PDO on X of order $m > 0$ ($m=1$, say). Then $P \boxtimes 1$ belongs to the closure of PDOs of order m on $X \times Y$ in the operator norm for operators $W^m \rightarrow W^0 = L^2$, and its symbol is $\sigma_P \boxtimes 1$.

(7)

Proof Let $p(x, \xi)$ be the symbol of P (we are now working in a local co-ordinate chart). Use variables $(x, \xi), (y, \eta)$.

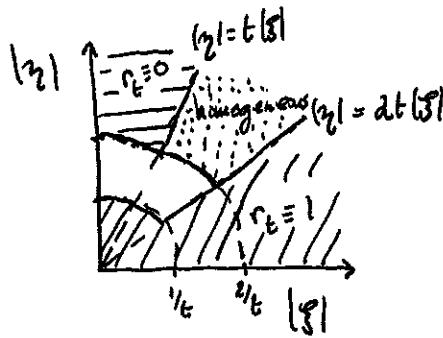
Consider a family of symbols

$$P_t(x, y, \xi, \eta) = p(x, \xi) \underbrace{r_t(\xi, \eta)}_{\text{bump}}$$

such that

(a) P_t is a genuine symbol $\forall t > 0$

(b) $P_t \rightarrow P$ as $t \rightarrow 0$ strongly enough to get norm convergence.



It's all in how we ~~so~~ construct the function $r_t(\xi, \eta) = r_t(|\xi|, |\eta|)$.

Proof of property (a) We ~~so~~ must estimate the derivatives $\partial_x^\beta p_t$ (the x derivatives ~~do not~~ are not a problem as r_t is just a bump). So ~~so~~ $\partial_x^\beta [p(x, \xi) r_t(\eta, \xi)]$ must be estimated.

Now, on $\text{supp}(r_t)$, $|\eta| \leq \frac{1}{t} \max\{2, |\xi|\}$
 $\Rightarrow (1 + |\xi| + |\eta|) \leq C_t (1 + |\xi|)$

So an estimate

$$\begin{aligned} |\partial_x^\beta p_t| &\leq C(C(1 + |\xi|))^{-k} \\ \Rightarrow |\partial_x^\beta p_t| &\leq C_t (1 + |\xi|)^{-k}. \end{aligned}$$

Proof of property (b) Look at $\|P_t - P\|_{op}$. Want to show that

$$(*) \sup \left| \frac{\partial_x^m p(x, z)(1 - r_t(z, y))}{(1 + |z| + |y|)^m} \right| \rightarrow 0 \text{ as } t \rightarrow 0.$$

For $t \rightarrow 0$, consider the region outside $r_t = 1$, so $|y| \geq 2t^{-1}|z|$ & $|y_1|, |z_1| \geq \frac{1}{2t}$. Thus

$$\frac{|p(x, z)|}{(1 + |z|)^m} \cdot \frac{(1 + |z|)^m}{(1 + |z| + |y|)^m} \geq (*)$$

and $\frac{p(x, z)}{(1 + |z|)^m}$ is bounded, and $\frac{(1 + |z|)^m}{(1 + |z| + |y|)^m}$ is $O(t^m)$.

$$\Rightarrow (*) \leq \underbrace{\frac{|p(x, z)|}{(1 + |z|)^m}}_{\text{bounded}} \cdot \frac{(1 + |z|)^m}{(1 + |z| + |y|)^m} \underset{O(t^m)}{\longrightarrow} 0.$$