

# Elliptic Operators and Topology

Lecture 26, 2014.12.4

Atiyah-Singer I.

Let  $D$  be elliptic operator on compact  $M$ , then  $[\sigma(D)] \in K(TM)$ .

$M \xrightarrow{i} S^n \xleftarrow{j} pt$  induces

$$K(TM) \xrightarrow{i_!} K(TS^n) \xleftarrow{j_!} K(Tpt) = \mathbb{Z}.$$

$\swarrow$

This is index.

Atiyah-Singer III.

$$\text{Index}(D) = \pm \int_{TM} ch(\sigma(D)) \cup td(TM \otimes \mathbb{C}).$$

- Relationship?
- Thom isomorphism in  $K$ -theory.

Let  $E$  be a complex vector bundle over  $X$ .  $E \xrightarrow[\sim]{i^*} X$

Thom isomorphism  $\varphi_E : K(X) \longrightarrow K(E)$

$$u \longmapsto u \cdot (\text{Thom class}).$$

$$\text{Thom class: } p^* \Lambda^0(E) \longrightarrow p^* \Lambda^1(E) \longrightarrow p^* \Lambda^2(E) \longrightarrow \dots,$$

if restricted to each fiber, is the Bott generator.

In cohomology, let  $E \rightarrow X$  be oriented real vector bundle with dimension  $n$ , then Thom class  $t_E \in H_c^n(E)$ , s.t. its restriction to each fiber is the generator of  $H_c^n(\mathbb{R}^n) \cong \mathbb{Z}$ .

$$\text{Thom isomorphism } \psi_E : H_c^*(X) \longrightarrow H_c^{*+n}(E).$$

$$u \longmapsto u \cdot t_E$$

$i : X \longrightarrow E$  induces (assume  $X$  is compact).

$$H_c^n(E) \xrightarrow{i^*} H^n(X)$$

$$t_E \longmapsto e(E). \text{ (Euler class).}$$

- Chern classes.

associated to complex vector bundle  $E$ .

- $c_k(E) \in H^{2k}(X)$ .
- $c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$ , where  $c(E) = 1 + c_1(E) + c_2(E) + \dots$
- $c_1(E \otimes F) = c_1(E) + c_1(F)$  for  $E, F$  line bundles.
- $c_1(E) = e(E)$ , if  $E$  is a complex line bundle.

Note: If  $E$  is a general  $\mathbb{C}^n$  bundle, then  $c_n(E) = e(E)$ .

- Additive genus.

Let  $f(z)$  be an entire function. Additive genus associated to  $f$  is defined as follows. If  $E$  is a complex vector bundle, say

$E = E_1 \oplus E_2 \oplus \dots$ , then additive genus =  $\sum f(x_i)$ , where

$$x_i = c_1(E_i).$$

- Example :  $f(z) = e^z$ .

The additive genus is Chern character.

$$\begin{aligned} \text{ch}(E) &= e^{x_1} + e^{x_2} + \dots \\ &= n + (x_1 + x_2 + \dots) + \frac{1}{2}(x_1^2 + x_2^2 + \dots) + \dots \\ &= n + c_1(E) + \frac{1}{2}(c_1^2 - 2c_2) + \dots \end{aligned}$$

Proposition: i)  $\text{ch}(E \oplus F) = \text{ch}(E) + \text{ch}(F)$

ii)  $\text{ch}(E \otimes F) = \text{ch}(E) \cdot \text{ch}(F)$

iii) Thus, ch gives a ring homomorphism  $k(X) \rightarrow H^*(X; \mathbb{Q})$ .

- Multiplicative genus.

associated to f is  $\prod f(x_i)$ , where  $x_i = c_1(E_i)$ .

If  $E \xrightarrow{\quad P \quad} X$ , then  $k(X) \xrightarrow{\varphi_E} k(E)$

$$\begin{array}{ccc} & \downarrow \text{ch} & \downarrow \text{ch} \\ H^*(X) & \xrightarrow{\psi_E} & H_c^*(E) \end{array}$$

does not commute!

What is  $\mu(E) := \psi_E^{-1}(\text{ch} \circ \varphi_E(1))$ ?

- Proposition:  $\mu(E)$  is the multiplicative genus associated to the function

$$f(z) = \frac{1-e^z}{z}.$$

Proof: Consider  $e(E) \mu(E) = i^* \psi_E^{-1} \cdot \psi_E^{-1}(\text{ch}(\varphi_E(1)))$ .

$$\begin{aligned} &= i^* \psi_E^{-1}(\psi_E^{-1} \circ \text{ch} \circ \varphi_E(1)) \quad (\text{By definition of } \psi_E). \\ &= i^*(\text{ch} \circ \varphi_E(1)) \end{aligned}$$

$$= \text{ch}(i^*(\varphi_E(1))) \quad (\text{By naturality of characteristic class}).$$

But  $i^*\varphi_E(1) = \sum (-1)^j \Lambda^j(E) \in k(X)$ .

- If  $E$  is a line bundle,

$$\text{ch}(i^*\varphi_E(1)) = 1 - e^{c_1(E)}.$$

- Observe  $\text{ch}(i^*\varphi_E(1))$  is multiplicative, because  $\Lambda^*(E \oplus F) = \Lambda^*E \hat{\otimes} \Lambda^*F$ .

$$\text{So } \text{ch}(i^*\varphi_E(1)) = \prod (1 - e^{x_i}).$$

$$\text{So } \mu(E) = \prod \frac{1 - e^{x_i}}{x_i}.$$

- From  $M \longrightarrow S^N \longleftarrow pt$   
(or  $\mathbb{R}^N$ ).

$$K(TM) \longrightarrow K(T\mathbb{R}^N) \xleftarrow{\sim} K(Tpt)$$

$$\textcircled{n} = \mu(N \otimes \mathbb{C})$$

$$\downarrow \text{ch} \quad \textcircled{n} \quad \downarrow \text{ch} \quad ? \quad \downarrow \text{ch}$$

$\downarrow$   
normal bundle of  $M$  in  $\mathbb{R}^N$ .

$$H(TM) \longrightarrow H(T\mathbb{R}^N) \xleftarrow{\sim} H(pt)$$

$$\int \downarrow \quad ? \quad \int \downarrow \quad ? \quad \int \downarrow$$

$= \quad = \quad =$

$\int \downarrow$  integrate.

$$\text{Thus, index}(D) = \pm \int_{TM} \text{ch} \cup \mu(N \otimes \mathbb{C}).$$

$$\text{But } \mu(N)\mu(T) = \mu(N \oplus T) = 1$$

$$\text{So } \mu(N) = \mu(T)^{-1} \quad (\text{Todd class } \text{td}(TM \otimes \mathbb{C})).$$