Statistical Change-point Methods for Closed-loop Delay Estimation

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Abstract

The input-output delay of a process is an important parameter for control design and closed-loop identification. This paper proposes new methods for estimating the delay of a transfer function model in closed-loop based on statistical change-point detection methods. A Bayesian change-point approach, a sequential probability ratio test (SPRT), and a cumulative sum (CUSUM) approach are proposed and compared against a recently proposed Laguerre polynomial-based method. The first two methods assume that the correct orders of the closed-loop ARMA model description are known, while the last one (the CUSUM approach) does not make such assumption. Two simulation examples and a sensitivity analysis are presented for comparison purposes. It is shown that the SPRT method performs best overall and is easy to tune depending on the dynamical response of the process. It is also shown that a CUSUM applied to the observed measurements, which does not require a parametric model of the closed-loop process, performs better than existing methods.

Keywords: Change-point detection, Bayesian analysis, Closed-loop identification.

1 Introduction

In control design it is often required to identify a suitable transfer function model of the process from operating data collected while the process is adjusted by a feedback controller. This may

be necessary when open-loop identification of the process is not feasible because the process is open-loop unstable [18] or when the process changes with time thus model parameters must be re-estimated and the controller must be re-tuned periodically [2]. In this paper we propose new methods for closed-loop identification of the time-delay in a discrete-time transfer function model. The methods are based on statistical change-point detection techniques.

Knowledge of the input-output process delay is important in control design. Many optimal control design methods, such as minimum variance control [2], are sensitive to the knowledge of the process delay and the closed-loop process under a designed controller may become unstable if the model delay does not coincide with the process delay. Knowing the delay is also essential for the control performance indices that are used to compare the performance of a control loop to its best achievable performance and to decide whether it is warranted to re-tune a controller. In the controller performance indices proposed by Harris et. al. [14] and by Horch and Isaksson [16], the delay is the only process information that is required to be known.

Delay information is also crucial in closed-loop process identification. In closed-loop experiments the input is frequently a linear function of the output hence it does not persistently excite the process. As a consequence of this, the parameter estimates tend to be less precise and sometimes even a unique solution is not obtainable [18]. However, prior knowledge about the process model, such as knowing the delay, is often useful in improving the precision of the estimates of the remaining parameters. Vanli and Del Castillo [21] show how among some of the commonly available forms of prior process knowledge, the delay provides the greatest improvement in the quality of the models identified in closed-loop. For the design of minimum variance controllers, Bohlin [6] has shown that the parameters of such controllers can be uniquely estimated from closed-loop data as long as the process delay is known.

In this paper, it will be assumed that a proportional-integral (PI) controller is in operation while the delay estimation experiments take place. The controller will be assumed to at least be able to stabilize the process around a given target, but if delay information becomes available it could be re-tuned and perform better. A step response test is used to induce a change in the output by a change in the process target and a change-point detection method is applied to estimate the time of the output change. Unlike traditional system identification experiments, step response tests do not require frequent changes in the input. Change-point methods have been used frequently in the Time Series and Statistical Process Control (SPC) literature, however, their usefulness for input-output delay estimation has not been recognized.

System identification from step response data is commonly applied for tuning PI controllers, particularly in conjunction with the Ziegler-Nichols tuning methods [3]. However, these methods are usually tailored for low noise and open-loop systems. In the proposed approach we consider closed-loop identification for systems with significant noise that can be modeled with Box-Jenkins or autoregressive-moving average-exogenous (ARMAX) transfer function models.

In this paper, we propose new closed-loop delay estimation methods that are based on Bayesian, Sequential Probability Ratio Test (SPRT) and Cumulative Sum (CUSUM) change-

point detection techniques. We note that, even though the procedures are presented for closed-loop estimation, the results can be applied in a straightforward way to an open-loop process where the output change is induced by a change in the controllable factor. The remainder of the paper is organized as follows. In Section 2 the process assumptions are introduced. The Bayesian, SPRT, and CUSUM delay estimation approaches are described in Sections 3, 4 and 5, respectively. In Sections 6 and 7, a simulation example is presented to illustrate the approaches. In Section 8 the approaches are compared against the Laguerre method of Isaksson et. al. [17]. In Section 9 the sensitivity of the SPRT and the Laguerre methods is studied with respect to the choice of their tuning parameters.

2 Process Assumptions

Consider a process that can be described by the discrete-time transfer function model:

$$y_t = G(z^{-1})u_{t-k} + v_t \tag{1}$$

where y_t is the process output, u_t is the control input, $G(z^{-1})$ is a n_g order linear polynomial in the back shift operator z^{-1} (i.e., $z^{-1}y_t = y_{t-1}$) and v_t is a disturbance process. The process delay, denoted by k, is the number of whole time periods that elapses between a change is made in the control input and its effect is observed at the process output. By definition $k \geq 1$.

It will be assumed that the disturbance can be represented by an integrated moving average (IMA) Time Series model $v_t = (1 - z^{-1})^{-\alpha} H(z^{-1})\epsilon_t$ where $H(z^{-1})$ is an n_h order polynomial, α is the degree of integration and $\{\epsilon_t\}$ is a white noise process with variance σ^2. The process is assumed to be adjusted by the PI controller

$$u_t = \frac{D(z^{-1})}{F(z^{-1})}(y_t - d_t) \tag{2}$$

where d_t is the set-point, or target, of the process at time t. The controller polynomials are $F(z^{-1}) = 1 - z^{-1}$ and $D(z^{-1}) = c_1 + c_2 z^{-1}$, where c_1 and c_2 are the controller parameters. As mentioned before it is assumed that the process output y_t is on target under the actions of this controller.

The delay will be estimated during the closed-loop operation of the process. The closed-loop equation of the process can be obtained by substituting controller (2) in the process model (1). This results in the following autoregressive-moving average ARMA(p, q) time series model:

$$\begin{array}{ccccccc} \left(F(z^{-1}) - z^{-k}G(z^{-1})D(z^{-1})\right) y_t & = & -G(z^{-1})D(z^{-1})d_{t-k} & + & F(z^{-1})H(z^{-1})(1 - z^{-1})^{-\alpha}\epsilon_t \\ & & \text{or} & & \\ \phi(z^{-1})y_t & = & \xi_t & + & \theta(z^{-1})\epsilon_t \end{array} \tag{3}$$

where ξ_t is the intercept, $\phi(z^{-1}) = 1 - \phi_1 z^{-1} - ... - \phi_p z^{-p}$ is the AR polynomial and $\theta(z^{-1}) = 1 - \theta_1 z^{-1} - ... - \theta_q z^{-q}$ is the MA polynomial. Correct determination of the orders of the AR and

MA polynomials can be obtained by standard time series methods applied to the closed-loop data y_t (see [8], pp. 197-202). As noted earlier, for the purpose of estimating the delay, the set-point is changed in a step fashion, namely:

$$d_t = \begin{cases} 0 & t = 1, 2, ..., s \\ \delta & t = s+1, s+2, ..., n \end{cases} \tag{4}$$

where it is assumed, without loss of generality, that the initial set-point is zero. Here, δ is the new set-point, s is the time instant at which the set-point is changed and n is the number of observations. We will refer to the act of varying d_t over n periods as the *delay estimation experiment*.

The set-point change δ is specified according to desired step shift size at the output. In a process controlled by a PI controller the output is able to track a constant set-point change exactly. Thus, letting μ_1 and μ_2 denote the initial and the final means of the process, respectively, we have that $\mu_2 - \mu_1 = \delta$. The step shift size can also be defined as $\mu_2 - \mu_1 = a\sigma_y$ where σ_y is the process standard deviation and a is the signal-to-noise ratio.

There is no additional delay in the process dynamics polynomial $G(z^{-1})$ or in the controller polynomial $D(z^{-1})$ because $G(0) \neq 0$ and $D(0) \neq 0$. It is clear, therefore, from (3) that the effect of the change made in the set-point d_t at time s is first observed in the output y_t at time $r = s + k$. Therefore, if the output change-point r can be detected (by using a change-point method), then the delay can be simply estimated from

$$k = r - s. \tag{5}$$

The delay estimation problem has been extensively studied in the control systems literature. Björklund and Ljung [5] give a comparison study of some of the recent open-loop delay estimation approaches. Isaksson, Horch and Dumont [17] proposed a closed-loop delay estimation method that uses a Laguerre transfer function model and step response tests. Laguerre functions, through their orthonormality properties, have the advantage of more accurately approximating transient process behavior and hence are suitable in estimating the delay in a step response test [22]. The next 3 sections present new delay estimation methods which will be contrasted with the Laguerre polynomial model of [17] in Sections 8 and 9.

3 Bayesian approaches

In Bayesian approaches, inferences about the parameters are based on the posterior distribution of the parameters, found by applying the well-known Bayes' rule:

$$\pi(\boldsymbol{\beta}|\boldsymbol{y}) \propto p(\boldsymbol{y}|\boldsymbol{\beta})\pi(\boldsymbol{\beta}) \tag{6}$$

where $\boldsymbol{\beta}$ is the vector of parameters, $p(\boldsymbol{y}|\boldsymbol{\beta})$ is the likelihood function, $\boldsymbol{y} = (y_1, y_2, ..., y_n)$ is the observed output data and $\pi(\boldsymbol{\beta})$ is the prior distribution which reflects the knowledge the user has

about the parameters before observing the data. The likelihood function of an autocorrelated sequence that obeys an ARMA model can be written as

$$p(\boldsymbol{y}|\boldsymbol{\beta}) = p(y_1|\boldsymbol{\beta})p(y_2|y_1,\boldsymbol{\beta})\ldots p(y_n|y_1,...,y_{n-1},\boldsymbol{\beta}), \tag{7}$$

that is, as the product of the conditional distributions of the observations given the past data and the parameters [19].

The change-point detection problem has been studied from a Bayesian viewpoint by many authors. Carlin, Gelfand and Smith [10] proposed a hierarchical Bayesian model for independent sequence of observations where inference about the change point was made by simulating the joint posterior of all model parameters using Gibbs sampling. Change-point detection in ARMA processes was discussed in Booth and Smith [7] where the autocorrelated sequence was transformed to an independent sequence to conduct the analysis. Bayesian analysis of time series models has been studied, among others, by Marriott et. al. [19] who used Markov chain Monte Carlo (MCMC) methods to simulate the posterior distribution of the ARMA parameters. We will utilize the results of [10] and [19] for estimating the input-output delay.

In processes that exhibit dynamic behavior the effect of the change made in the set-point is fully realized at the output after several subsequent time periods. This is in sharp contrast to the usual assumption made in change-point methods that the output change is fully realized in one time period. In order to account for the dynamic behavior of the process the mean step-response of the process can be approximated by the following exponential rise model:

$$\mu_t = E(y_t) = \begin{cases} \mu_1 & t = 1,...,r \\ \mu_1 + (\mu_2 - \mu_1)\left(1 - e^{-\frac{t-r}{\gamma}}\right) & t = r+1,...,n \end{cases} \tag{8}$$

where γ is the time constant of the process which is the time required for the process to rise to 63% of its steady-state value. By definition $\gamma \geq 0$. It can be shown that expression (8) is the step response of a first order transfer function model ([8], p. 376).

First order models are commonly used to approximate the dynamic process behavior in automatic process control [3]. A second order transfer function may also behave close to a first order process when it has a non-oscillatory step response. Then, the model given in equation (8) can be used to approximate its behavior. Delay estimation under a second order process is discussed further in Section 7.

In order to derive the likelihood function we represent the ARMA model of equation (3) in the following mean centered form:

$$\begin{aligned} \phi(z^{-1})(y_t - \mu_t) &= \theta(z^{-1})\epsilon_t \\ &\text{or} \\ y_t - \mu_t - \tau_t &= \epsilon_t \end{aligned} \tag{9}$$

where μ_t is the mean of the process at time t represented by the exponential rise model (8) and

τ_t is

$$\tau_t = \sum_{i=1}^{p} \phi_i \left(y_{t-i} - \mu_{t-i}\right) - \sum_{j=1}^{q} \theta_j \epsilon_{t-j}, \qquad t = 2, 3, ..., n \tag{10}$$

and $\tau_1 = 0$. This can be evaluated recursively by replacing the unobservable errors ϵ_t with the residuals of the model (9) as:

$$\tau_t = \sum_{i=1}^{p} \phi_i \left(y_{t-i} - \mu_{t-i}\right) - \sum_{j=1}^{q} \theta_j (y_{t-j} - \mu_{t-j} - \tau_{t-j}). \tag{11}$$

Therefore, by using (9) and assuming that $\epsilon_t \overset{iid}{\sim} N(0, \sigma^2)$ we get

$$p(y_1|\boldsymbol{\beta}) = N(\mu_1, \sigma^2), \tag{12}$$

$$p(y_t|y_1, ..., y_{t-1}, \boldsymbol{\beta}) = N(\mu_t + \tau_t, \sigma^2), \qquad t = 2, 3, ..., n. \tag{13}$$

We also note that the vector of all the parameters in this model is

$$\boldsymbol{\beta} = (\mu_1, \mu_2, \boldsymbol{\phi}, \boldsymbol{\theta}, \sigma^2, r, \gamma)$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)$ and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_q)$. The multiplication of (12) and (13) gives the likelihood function as

$$\begin{aligned} p(\boldsymbol{y}|\boldsymbol{\beta}) &= \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}\left(y_t - \mu_t - \tau_t\right)^2\right) \\ &= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} A\right) \end{aligned} \tag{14}$$

where $A(\boldsymbol{\phi}, \boldsymbol{\theta}, \mu_1, \mu_2, \gamma, r) = \sum_{t=1}^{n} \left(y_t - \mu_t - \tau_t\right)^2$. The standard ARMA model assumptions of stationarity and invertibility need to be enforced in the likelihood. Let $\mathcal{S}_\phi$ denote the set of $\boldsymbol{\phi}$ values that satisfy the stationarity condition (i.e., all roots of $\phi(z^{-1})$ must lie outside the unit circle) and let $\mathcal{S}_\theta$ denote the set of $\boldsymbol{\theta}$ values that satisfy the invertibility condition (i.e., all roots of $\theta(z^{-1})$ must lie outside the unit circle); we therefore assume the likelihood (14) to be valid only for $\boldsymbol{\phi} \in \mathcal{S}_\phi$ and $\boldsymbol{\theta} \in \mathcal{S}_\theta$.

3.1 Prior specification of the parameters

Let $\tilde{\boldsymbol{\beta}} = (\mu_1, \mu_2, \boldsymbol{\phi}, \boldsymbol{\theta}, \sigma^2)$ denote the vector of ARMA model parameters, thus $\boldsymbol{\beta} = (\tilde{\boldsymbol{\beta}}, r, \gamma)$. Assuming that the parameters are independent a priori, the joint prior of all the model parameters is

$$\pi(\boldsymbol{\beta}) = \pi(\tilde{\boldsymbol{\beta}})\pi(r)\pi(\gamma). \tag{15}$$

It will be assumed that no prior knowledge about the ARMA model parameters exists. The usual noninformative priors for these parameters are ([12] pp. 61-62):

$$\begin{aligned} \pi(\mu_1, \mu_2, \boldsymbol{\phi}, \boldsymbol{\theta}) &\propto \text{constant}, \\ \pi(\sigma^2) &\propto \frac{1}{\sigma^2}. \end{aligned}$$

Therefore,

$$\pi(\tilde{\boldsymbol{\beta}}) \propto \frac{1}{\sigma^2}. \tag{16}$$

Prior knowledge about the delay k and the time constant γ may be available from prior experience with the process. In most industrial processes the delay values are typically less than a few time intervals and even when precise knowledge is not available one can usually set a sensible upper bound on the possible values of k. For example, if k_0 is a prior estimate of the upper bound of the delay then by using (5) an upper bound on the change point r is $s + k_0$. Thus, a prior on r for this case is a Discrete Uniform distribution:

$$\pi(r) = DU(s+1, s+k_0), \qquad r = s+1, s+2, ..., s+k_0$$

Here, only the values $r \geq s+1$ are possible because by definition the delay is greater than 1. More precise prior knowledge about k can be represented by choosing a Binomial distribution as a more informative prior:

$$\pi(r) = Bin(k_0, p_0, s) = \begin{pmatrix} k_0 - 1 \\ r - s - 1 \end{pmatrix} p_0^{r-s-1}(1-p_0)^{k_0-(r-s)}, \qquad r = s+1, s+2, ..., s+k_0$$

where, similarly to the uniform prior, k_0 is an upper bound on k. The probability of success p_0 can be selected so that the mode of the prior coincides with a prior estimate of r.

Prior knowledge about the time constant can be obtained from observing the process during a step response test. A common informative prior for non-negative continuous parameters is a Log-Normal distribution:

$$\pi(\gamma) = LN(\mu_\gamma, \sigma_\gamma^2), \qquad \gamma \geq 0.$$

By definition of a log-normal, $\gamma^* = \log \gamma$ is distributed as $N(\mu_\gamma, \sigma_\gamma^2)$. Therefore, the location parameter μ_γ can be set equal to a prior estimate of γ^* and the scale parameter σ_γ^2 can be adjusted to control the confidence on the prior. When no prior knowledge about the time constant exists, a non-informative prior is assumed on γ:

$$\pi(\gamma) \propto \frac{1}{\gamma}, \qquad \gamma \geq 0.$$

3.2 Posterior distribution of the parameters

From (6) we obtain the posterior distribution of the parameters:

$$\pi(\boldsymbol{\beta}|\boldsymbol{y}) \propto (\sigma^2)^{-\frac{n}{2}-1} \exp\left(-\frac{1}{2\sigma^2}A\right) \pi(r)\pi(\gamma) \tag{17}$$

which is defined for $\boldsymbol{\phi} \in \mathcal{S}_\phi$ and $\boldsymbol{\theta} \in \mathcal{S}_\theta$ (from the stationarity and the invertibility conditions of the likelihood), and $\gamma \geq 0$ (from the prior specification). In particular, we are interested in the marginal posterior of the change-point r. The change-point is estimated as the mode of this distribution, and the delay k is estimated by using (5).

The unnormalized posterior density of $\boldsymbol{\beta}$ given in (17) does not correspond to any standard multivariate density function hence a closed-form expression of it is not available. The posterior of r will be obtained numerically by considering the following two cases: (i) All model parameters are assumed unknown hence the joint posterior (17) needs to be obtained. This will be referred to as the *full* Bayesian approach; (ii) The ARMA model parameters $\tilde{\boldsymbol{\beta}}$ are assumed known hence only the posterior of (r, γ) needs to be obtained. This will be referred to as the *reduced* Bayesian approach.

3.3 Full Bayesian Approach

The joint posterior (17) was simulated by using Markov Chain Monte Carlo (MCMC) methods [13]. In sampling with MCMC methods it is recommended to find a suitable variable transformation to obtain an unconstrained density in order to improve the speed of convergence. For the variables $\gamma, \boldsymbol{\phi}$ and $\boldsymbol{\theta}$ that have constrained supports in the joint posterior density (17) we employed variable transformations [19].

The time constant γ was transformed as $\gamma^* = \log \gamma$ so that $\gamma^* \in \mathbb{R}$ is unconstrained. The Jacobian of the transformation is $J(\gamma \to \gamma^*) = e^{\gamma^*}$. The AR and MA parameters $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ were transformed to $\boldsymbol{\phi}^* \in \mathbb{R}^p$ and $\boldsymbol{\theta}^* \in \mathbb{R}^q$ by applying the transformations recommended in [19]. We refer to this paper for the required transformations, the corresponding Jacobians $J(\boldsymbol{\phi} \to \boldsymbol{\phi}^*)$ and $J(\boldsymbol{\theta} \to \boldsymbol{\theta}^*)$ and the inverse transformations.

Let $\boldsymbol{\beta}^* = (\mu_1, \mu_2, \boldsymbol{\phi}^*, \boldsymbol{\theta}^*, \sigma^2, r, \gamma^*)$ denote the vector of transformed parameters. The joint posterior of $\boldsymbol{\beta}^*$ is obtained as:

$$\pi(\boldsymbol{\beta}^*|\boldsymbol{y}) \propto (\sigma^2)^{-\frac{n}{2}-1} \exp\left(-\frac{1}{2\sigma^2}A\right) \pi(r)\pi(\gamma^*)e^{\gamma^*} J(\boldsymbol{\phi} \to \boldsymbol{\phi}^*)J(\boldsymbol{\theta} \to \boldsymbol{\theta}^*). \tag{18}$$

To simulate the posterior we divided $\boldsymbol{\beta}^*$ into the sub-blocks $\mu_1, \mu_2, \boldsymbol{\phi}^*, \boldsymbol{\theta}^*, \sigma^2, r, \gamma^*$ and sampled each sub-block from its conditional posterior distribution given all the other sub-blocks. The blocks are sampled in the given order and an iteration is completed when all sub-blocks are sampled. The sub-blocks are sampled using the Metropolis-Hastings algorithm. At the end of each iteration, the sampled $\boldsymbol{\beta}^*$ value is converted to $\boldsymbol{\beta}$ by applying inverse transformations.

A single Markov chain was simulated to generate the posterior distribution. The chain was started from a randomly selected starting point $\boldsymbol{\beta}^{(0)}$ and was run for $I = 25000$ iterations out of which the initial $I_0 = 4999$ burn-in iterations were discarded to eliminate the biasing effect of the starting point. The Raftery and Lewis convergence diagnostic tool ([13], pp. 115-130) was used to verify that the adopted burn-in length was adequate and the chain has converged to its stationary distribution within the adopted chain length. The generated samples $\boldsymbol{\beta}^{(i)}, i = I_0, ..., I$ were used as the posterior distribution. See [20] for the implementation of the MCMC algorithm and the convergence diagnostic tool.

3.4 Reduced Bayesian Approach

The Bayesian formulation can be simplified considerably by assuming that the ARMA model parameters $\tilde{\boldsymbol{\beta}}$ are known. The ARMA model parameters can often be estimated independently of the change point (see Appendix A.2).

Since $\tilde{\boldsymbol{\beta}}$ is assumed known, the only unknown parameters in this case are r and γ. The joint posterior of (r, γ) conditional on the observed data and on $\tilde{\boldsymbol{\beta}}$ can be obtained from (6) as:

$$\pi(r, \gamma|\boldsymbol{y}, \tilde{\boldsymbol{\beta}}) \propto p(\boldsymbol{y}|r, \gamma, \tilde{\boldsymbol{\beta}})\pi(r, \gamma|\tilde{\boldsymbol{\beta}}) \tag{19}$$

and, by assuming prior independence, we have that $\pi(r, \gamma|\tilde{\boldsymbol{\beta}}) = \pi(r)\pi(\gamma)$. By substituting the likelihood (14) in (19) and applying the transformation $\gamma^* = \log\gamma$, the posterior of (r, γ^*) is:

$$\begin{aligned} \pi(r, \gamma^*|\boldsymbol{y}, \tilde{\boldsymbol{\beta}}) &\propto \exp\left(-\frac{1}{2\sigma^2}A(r, \gamma^*)\right)\pi(r)\pi(\gamma^*)e^{\gamma^*} \qquad (20) \\ &= h(r, \gamma^*|\boldsymbol{y}, \tilde{\boldsymbol{\beta}}) \end{aligned}$$

where $h(r, \gamma^*|\boldsymbol{y}, \boldsymbol{\beta})$ is the unnormalized posterior and A is defined in the same way as in Section 3. The marginal posterior of r is obtained by integrating (20) with respect to γ^*:

$$\begin{aligned} \pi(r|\boldsymbol{y}, \tilde{\boldsymbol{\beta}}) &\propto \int_{-\infty}^{\infty} h(r, \gamma^*|\boldsymbol{y}, \tilde{\boldsymbol{\beta}})d\gamma^* \qquad (21) \\ &= h(r|\boldsymbol{y}, \tilde{\boldsymbol{\beta}}), \qquad r = s+1, s+2, ..., s+k_0 \qquad (22) \end{aligned}$$

where $h(r|\boldsymbol{y}, \tilde{\boldsymbol{\beta}})$ is the unnormalized marginal posterior of r and can be computed from (21) by using Monte Carlo integration. It has the same mode as $\pi(r|\boldsymbol{y}, \tilde{\boldsymbol{\beta}})$ thus can be directly used for delay estimation. We also note that in this case, unlike the full Bayesian analysis, an MCMC simulation is not required to compute the posterior probabilities of r.

4 A sequential probability ratio test approach

The problem of change-point detection commonly arises in industrial process monitoring. Sequential probability ratio tests (SPRT) are commonly employed for change-point detection in correlated data [4]. In SPRT the change point is estimated by conducting a series of hypothesis tests where the null hypothesis $H_0 : \mu = \mu_1$ is tested against the alternative hypothesis $H_1 : \mu = \mu_1 + R$. Here, R is a reference value selected as the magnitude of the shift (step) to be detected. In this study the reference value is specified as a multiple of the process standard deviation by setting $R = b\sigma_y$ where $b < a$ and b is the reference value in the standard deviation units. Recall from Section 2 that a is the actual shift size in the standard deviation units (signal-to-noise ratio). Appendix A.1 describes how b can be selected according to the dynamics of the process.

A log-likelihood ratio statistic, S_m, computed at each time instant m by assuming an ARMA model is fit to the closed-loop process, is used as the decision rule of the hypothesis test. The

typical behavior of the log-likelihood ratio as a function of time shows a negative slope before the change and a positive slope after the change. Therefore, the point at which this statistic reaches a minimum is used as an estimate of the change point ([4], p. 41).

To estimate the output change-point that is caused by the set-point change at a time instant s, it is sufficient to search for the minimum only among the time periods $m = s+1, s+2, ..., n$ since by definition the delay is greater than 1. Therefore, it is sufficient to compute the likelihood of the sequences $\{y_t\}_{t=s+1}^{m}$ for the time periods $m = s+1, s+2, ..., n$.

Let $\mu^{(0)} = \mu_1$ and $\mu^{(1)} = \mu_1 + R$ denote the mean of the process under the null and alternative hypotheses, respectively. The log-likelihood ratio criterion for testing the hypothesis is therefore defined as:

$$S_m = \ln \frac{p(y_{s+1}, ..., y_m|\mu^{(1)})}{p(y_{s+1}, ..., y_m|\mu^{(0)})}. \tag{23}$$

The likelihood under each hypothesis can be obtained by applying (7):

$$p(y_{s+1}, ..., y_m|\mu^{(l)}) = p(y_{s+1}|\mu^{(l)}) \dots p(y_m|y_{s+1}, ..., y_{m-1}, \mu^{(l)}) \tag{24}$$

where $l = 0$ and $l = 1$ denote the null and the alternative hypotheses, respectively. Here, by using (13), the conditional distributions of the observations are

$$p(y_t|y_{s+1}, ..., y_{t-1}, \mu^{(l)}) = N(\mu^{(l)} + \tau_t^{(l)}, \sigma^2), \qquad t = s+1, s+2, ..., m \tag{25}$$

where, by applying (11), $\tau_t^{(l)}$ is

$$\tau_t^{(l)} = \sum_{i=1}^{p} \phi_i (y_{t-i} - \mu^{(l)}) - \sum_{j=1}^{q} \theta_j (y_{t-j} - \mu^{(l)} - \tau_{t-j}^{(l)})$$

As it can be noticed, there are several similarities between the likelihood definitions of the Bayesian and the SPRT approaches. One important difference, however, is that while the (full and reduced) Bayesian approaches assume the exponential function (8) as the mean, the SPRT approach assumes a constant mean $\mu^{(l)}$.

Substituting (25) in (24), the likelihood function under a given hypothesis is:

$$p(y_{s+1}, ..., y_m|\mu^{(l)}) = \prod_{t=s+1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}\left(y_t - \psi_t^{(l)}\right)^2\right)$$

where $\psi_t^{(l)} = \mu^{(l)} + \tau_t^{(l)}$. The log-likelihood ratio (23) can therefore be written as the cumulative sum

$$S_m = \sum_{t=s+1}^{m} V_t$$

where the increments of the sum are

$$V_t = \frac{1}{\sigma^2}\left(\psi_t^{(1)} - \psi_t^{(0)}\right)\left(y_t - \frac{\psi_t^{(1)} + \psi_t^{(0)}}{2}\right).$$

The point m at which S_m is minimum is the estimate of change point, and hence, of the delay using (5). In SPRT the parameters $\mu_1, \mu_2, \sigma^2, \boldsymbol{\phi}, \boldsymbol{\theta}$ need to be known (see Appendix A.2 for estimation details).

5 A cumulative sum approach

Cumulative sum (CUSUM) control charts are also commonly used in industrial process monitoring [15]. In this section two versions of the CUSUM scheme are presented: a CUSUM on the original process data (i.e., no ARMA modelling is necessary) and a CUSUM on the residuals of an ARMA time series model fitted to the original data. We refer also to Appendix A.2 for details regarding the estimation of the required parameters for this approach.

CUSUM on the original data

If considerable uncertainty about the orders of the ARMA model that describes the closed-loop operation of the process exists, the following CUSUM delay estimation method is suggested.

The application of the CUSUM statistic can be regarded as a series of hypothesis tests where the null hypothesis $H_0 : \mu = \mu_1$ is tested against the alternative hypothesis $H_1 : \mu = \mu_1 + K$. Here, K is a reference value that is usually set at one-half magnitude of the shift that we want to detect ([15], pp. 32). A CUSUM statistic C_m computed at time m is used as the decision rule of the hypothesis test.

A process monitored with a CUSUM control chart is considered to have gone out of control when the C_m statistic exceeds an upper control limit H (i.e. when $C_m > H$). Here, a recommended value for H is 4 times the process standard deviation σ_y (see e.g., [15]). Once an out-of-control signal is issued (say, at point m') our approach estimates the change-point as the first point, prior to this signal, at which the test statistic began raising above zero (i.e. when $C_m > 0$ where $m \leq m'$). Similarly to the SPRT approach, for delay estimation C_m needs to be monitored only for the time periods $m = s + 1, s + 2, ..., n$.

We utilize a one-sided CUSUM procedure where it is desired to detect a change in a given direction given by the new set-point. We will assume here, without loss of generality, that there is an increase in the output level and hence a one-sided upper CUSUM can be used to detect the change-point. An upper CUSUM can be computed as:

$$C_m = \begin{cases} 0 & m = s+1 \\ \max\left(0, y_m - (\mu_1 + K) + C_{m-1}\right) & m = s+2, ..., n. \end{cases} \tag{26}$$

Similarly to the SPRT approach, the reference value K is specified as a multiple of the process standard deviation by setting $K = \frac{b}{2}\sigma_y$ (one-half of R) where b is selected according to the dynamics of the process (see Appendix A.1).

CUSUM on the residuals

In processes with strong autocorrelation, monitoring the original data may lead to frequent false alarms. To avoid this problem, an alternative CUSUM approach widely studied in the SPC literature is to use a CUSUM to monitor the residuals of an ARMA time series model fitted to the data [1].

According to the ARMA(p,q) model (3), the fitted output value $\hat{y}_t$ calculated at time $t-1$ using the past data $(y_1,...,y_{t-1})$ and the parameters $(\boldsymbol{\phi},\boldsymbol{\theta},\mu_1)$ is

$$\hat{y}_t = \phi_1 y_{t-1} + \ldots + \phi_p y_{t-p} + \xi - \theta_1 e_{t-1} - \ldots - \theta_q e_{t-q} \tag{27}$$

where $e_t = y_t - \hat{y}_t$ is the model residual and $\xi = \mu_1(1-\phi_1-...-\phi_p)$ is the intercept. Note that according to this model, the predictions are computed with respect to the initial mean μ_1 of the process and the mean change to μ_2 is ignored. This decreases the degree of filtering in the step shift (as opposed to being filtered out) which in turn makes detection easier. The CUSUM statistic (26) for the residuals is calculated by replacing y_m, μ_1, μ_2 and σ_y with e_m, μ_{1e}, μ_{2e} and σ_e, respectively. The reference value of the detection rule is set as $K_e = \frac{b}{2}\sigma_e$.

6 Example 1: Illustration of the proposed methods

To illustrate the three proposed methods first consider a simple pure gain transfer function with additive IMA(1,1) noise, commonly employed in the control of discrete-part manufacturing processes [9]:

$$y_t = g u_{t-k} + \frac{1-\theta z^{-1}}{1-z^{-1}}\epsilon_t. \tag{28}$$

where the delay is $k=3$, the parameters are $g=2$ and $\theta=0.3$ and the white noise process $\{\epsilon_t\}$ follows a normal distribution with 0 mean and variance 1. The true values of the model parameters are unknown.

Suppose the process is controlled with a PI controller $u_t = \frac{c_1+c_2 z^{-1}}{1-z^{-1}}(y_t - d_t)$ where $(c_1,c_2) = (-0.25, 0.15)$. By using (3) the closed-loop model of this process under the actions of the PI controller is:

$$(1 - z^{-1} - g c_1 z^{-3} - g c_2 z^{-4}) y_t = -g(c_1 + c_2 z^{-1}) d_{t-3} + (1-\theta z^{-1})\epsilon_t \tag{29}$$

which is an ARMA(4,1) process with parameters $\boldsymbol{\phi} = (1, 0, -0.5, 0.3)$ and $\theta = 0.3$.

The delay estimation approaches were applied on the simulated delay estimation experiments during which the process was adjusted with the PI controller and the set-point was increased by $\delta = 3\sigma_y$ and $\delta = 6\sigma_y$ units. Two shift (step) sizes were used in order to evaluate the performance under a small and a large shift. The standard deviation of this process is $\sigma_y = 1.45$. The length of the experiment and the set-point change time were $n = 200$ and $s = 100$, respectively. Since the input-output delay is 3 periods, the true change-point is $r = 103$. The simulation was repeated 100 times. This was limited due to the long computing time of the full Bayesian approach; see below for the CPU times. For more extensive simulations of all other approaches see Section 8. Figure 1 shows a simulation of the process and the corresponding residuals obtained by fitting an ARMA(4,1) model to this realization.

Both the full and the reduced Bayesian approaches were applied. In both approaches a non-informative prior and an informative prior were assumed for the change point r and only

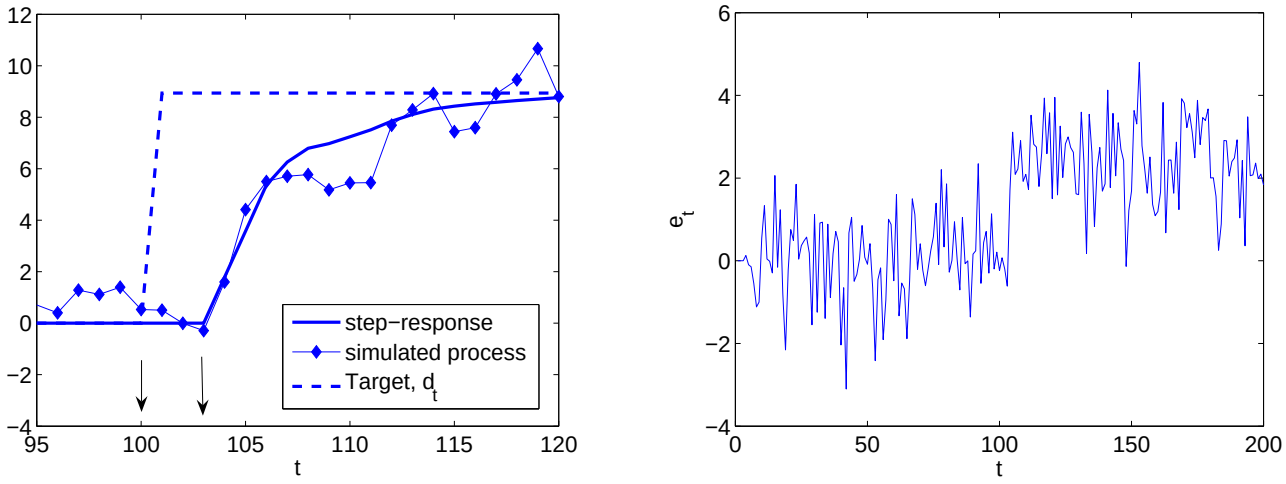


Figure 1: The simulated delay estimation experiment of the process in Example 1 with a $6\sigma_y$ step shift. **Left panel**: the process output (only observations 95 to 120 are shown). **Right panel**: the residuals of the ARMA model fitted to the output data. In the left panel the first arrow indicates the instant at which the change is made in the set-point ($s = 100$) and the second arrow indicates the change-point of the output ($r = 103$). For comparison the mean step-response of the process (solid line) is also shown.

a non-informative prior was assumed for the remaining parameters. A discrete uniform distribution $DU(101, 110)$ was used as a non-informative prior on r and a binomial distribution $Bin(10, 2/9, 100)$ was chosen as a more informative prior. In both priors an upper bound $k_0 = 10$ was employed. In the binomial prior the probability of success is set at $p_0 = 2/9$ so that its mode is equal to the true change point 103.

The SPRT and the CUSUM detection rules were tuned to detect a shift that is one-third magnitude of the actual shift by setting the reference value $b = 1$ for the small shift and $b = 2$ for the large shift. In the Bayesian, the SPRT and the CUSUM (on residuals) approaches the correct closed-loop model ARMA(4,1) was assumed.

Figure 2a shows the simulated marginal posterior of r obtained with the full Bayesian approach and by assuming non-informative priors on all parameters. From the mode of this distribution the change-point was estimated as $\hat{r} = 103$. Figure 2b shows the SPRT and CUSUM statistics S_m and C_m. We see that S_m reaches its minimum at time instant 103, thus the SPRT estimate of the change point is $\hat{r} = 103$. Likewise, C_m (both on original data and on residuals) raises above zero at time instant 103 thus the CUSUM estimate of the change-point is $\hat{r} = 103$. Since the set-point was changed at time 100 the delay estimate from all methods is $\hat{k} = 3$. All results were obtained from the realization given in Figure 1.

The performance of the methods were compared on the basis of the root mean squared error

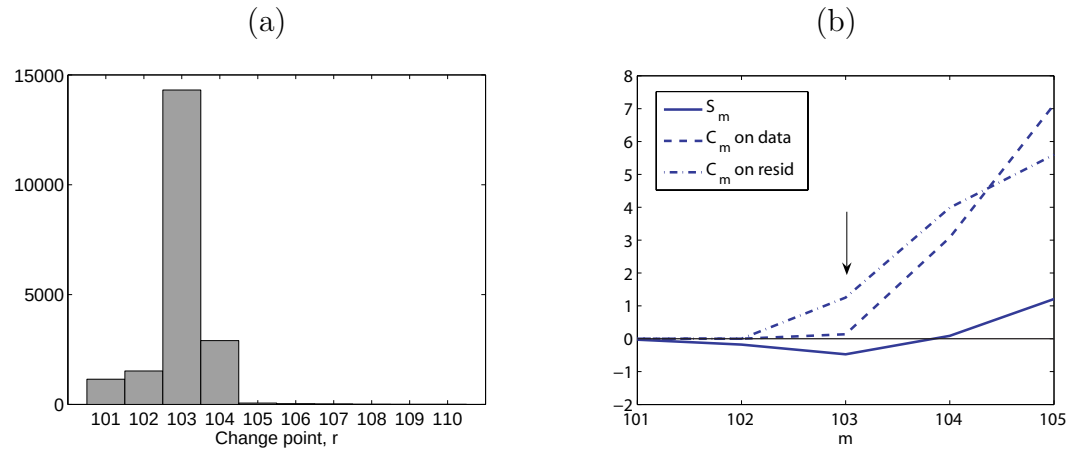


Figure 2: The results of the delay estimation analyses (Example 1, a $6\sigma_y$ step shift). (a) The histogram of the simulated posterior of the change point r using the full Bayesian analysis. (b) The S_m and C_m statistics of the SPRT and the CUSUM approaches. The arrow indicates the time instant at which the change is detected.

(RMSE) of the computed delay estimates. This is defined as:

$$RMSE = \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\hat{k}_i - k\right)^2} \tag{30}$$

where $\hat{k}_i$ is the delay estimate obtained from the ith realization, k is the true delay and $M = 100$ is the number of simulations.

Table 1 reports the mean, the variance and the RMSE of the delay estimates computed with the three methods. The average CPU time to compute one delay estimate with the SPRT, CUSUM, and the full and the reduced Bayesian approaches were 0.098, 0.093, 138.6 and 0.245 seconds, respectively (the computations were done on a 3.60GHz PC). The full Bayesian approach requires a significantly longer CPU time than the reduced Bayesian approach because it involves running a Markov Chain simulation to compute each delay estimate.

The SPRT and the CUSUM on residuals approaches provided similar RMSE performance. This is in agreement with the observation that SPRT is a generalization of CUSUM ([4], pg. 43). Also, since the process is autocorrelated, a CUSUM on the residuals provides a better performance than a CUSUM on the original data. The full Bayesian approach with an informative prior on r outperformed the alternatives under both shift sizes. By contrast, when no prior process knowledge exists the best delay estimates were obtained with the SPRT approach regardless of the shift size. It can also be seen that the reduced Bayesian approach provided comparable RMSE performance to the full Bayesian approach.

Shift Size		SPRT	CUSUM		Full Bayes		Reduced Bayes	
			On data	On residuals	Non-inf	Inf	Non-inf	Inf
$3\sigma_y$	Mean	2.67	3.07	2.75	4.26	3.31	4.56	3.37
	Var	1.86	2.53	2.07	4.86	1.10	4.71	1.10
	RMSE	**1.40**	**1.58**	**1.45**	**2.53**	**1.09**	**2.66**	**1.11**
$6\sigma_y$	Mean	2.72	3.34	2.69	3.23	3.11	3.15	3.15
	Var	0.81	1.12	0.80	0.87	0.46	0.88	0.51
	RMSE	**0.94**	**1.10**	**0.94**	**0.95**	**0.69**	**0.94**	**0.73**

Table 1: Mean, variance and RMSE of the delay estimates obtained with the SPRT, CUSUM and full and reduced Bayesian methods from 100 simulations (Example 1). The true delay is 3. "Non-inf" refers to the case where noninformative priors were used on all parameters and "Inf" refers to the case where an informative prior was used only on r and noninformative priors were used on the remaining parameters.

7 Performance of the proposed approaches under a second order process

In this section we illustrate delay estimation under a process that has an oscillatory step response. Consider again the process model given in equation (28). In this process an oscillatory closed-loop step-response was obtained by changing the PI controller parameters to $(c_1, c_2) = (-0.4, 0.2)$ from the original controller settings of $(c_1, c_2) = (-0.25, 0.15)$. Note that this is equivalent to an open-loop process with a second order transfer function. We will refer to the process considered in Section 6 under the original controller settings as the *first order process*, and the process considered in this section as the *second order process*.

Figure 3 gives a comparison of the step responses under the two controller settings. Only the SPRT approach (with $b = 1$) and the shift size $\delta = 3\sigma_y$ were considered. Table 2 gives the mean, variance and the RMSE of the delay estimates obtained from $M = 1000$ simulations under a second and a first order process.

	First Order	Second Order
Mean	2.78	2.53
Var	2.28	0.82
RMSE	1.53	1.02

Table 2: The mean, variance and the root mean square error (RMSE) of the delay estimates of the first order and second order processes obtained with SPRT. True delay is $k = 3$. Results are from 1000 simulations.

We observe that the delay estimates have better RMSE in a second order process than in a first order process. This is due to the fact that in an oscillatory second order process the

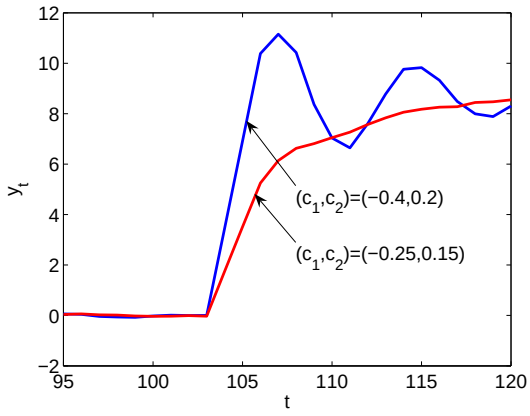


Figure 3: Comparison of the step responses of the process under the two PI controller settings. The experiment length is $n = 200$ and the step change in the set-point was introduced at time $s = 100$.

response first overshoots the set-point and hence has a shorter rise time (Figure 3) which makes change-point detection easier. Therefore, in general, a change-point detection method gives better delay estimates for a process with a second-order (oscillatory) step response then for a process with a first-order (exponential rise) step response.

8 Example 2: Comparison with the Laguerre method

In this section we compare the performance of the proposed delay estimation approaches to the Laguerre delay estimation method of Isaksson et. al. [17] by using two examples from [17]. The Laguerre delay estimation method is reviewed and illustrated by using these examples in Appendix A.3.

In the first process the delay is $k = 10$ and the dynamics and disturbance models are given by

$$G(z^{-1}) = \frac{0.0178 + 0.0123z^{-1}}{1 - 1.273z^{-1} + 0.333z^{-2}} \qquad \text{and} \qquad v_t = \frac{0.71}{1 - 0.7z^{-1}}\epsilon_t, \tag{31}$$

respectively, where $\epsilon_t \overset{iid}{\sim} N(0, 0.05^2)$. The process is controlled with the PI controller $u_t = -\frac{0.612-0.6z^{-1}}{1-z^{-1}}(y_t - d_t)$. By inserting this controller in (1) the closed-loop equation follows an ARMA(13,3) process. As pointed out by the authors [17], for this process estimating the delay is hard due to the slow dynamics despite the low noise.

In the second process the delay is also $k = 10$. The dynamics and the disturbance models are

$$G(z^{-1}) = \frac{0.296 + 0.0204z^{-1}}{1 - 0.368z^{-1} + 1.671 \times 10^{-5}z^{-2}} \text{ and } v_t = \frac{0.92}{1 - 0.368z^{-1} + 1.671 \times 10^{-5}z^{-2}}\epsilon_t, \tag{32}$$

respectively, where $\epsilon_t \overset{iid}{\sim} N(0, 0.1^2)$. The process is controlled with the PI controller $u_t = -\frac{0.3-0.25z^{-1}}{1-z^{-1}}(y_t - d_t)$. The closed-loop equation in this case is an ARMA(12,1) process. For this

process delay estimation is easier given the faster dynamics. In the proposed approaches that require an ARMA model of the closed-loop process, the correct model orders were assumed for the two processes. As pointed out earlier, this is not a strong assumption since the correct orders of the closed-loop ARMA model can be identified using standard time-series techniques applied to y_t.

In the simulations the PI controllers were adjusting the process, the length of the experiment was $n = 2000$ and the set-point was increased by $\delta = 2$ units at time $s = 1000$ (note that this is the same experimental condition employed in [17]). For both processes the delay is $k = 10$, thus the true change-point is $r = 1010$. The simulation was repeated 1000 times.

Dividing the step size δ by the process standard deviation we find the signal-to-noise ratio used in the experiment as $a = 40$ and $a = 20$ for the first and the second processes, respectively. It can be seen that compared to the signal-to-noise ratios ($a = 3$ and $a = 6$) employed in Example 1, these values are considerably large.

One simulated realization of each process is shown in Figure 4. Notice that the first process has a larger time constant (slower dynamics) than the second process. From the rise times the time constant can be approximated as $\gamma \approx 140$ for the first process and as $\gamma \approx 30$ for the second process. The corresponding $\gamma^* = \log \gamma$ values are 4.9 and 3.4 for the first and second processes.

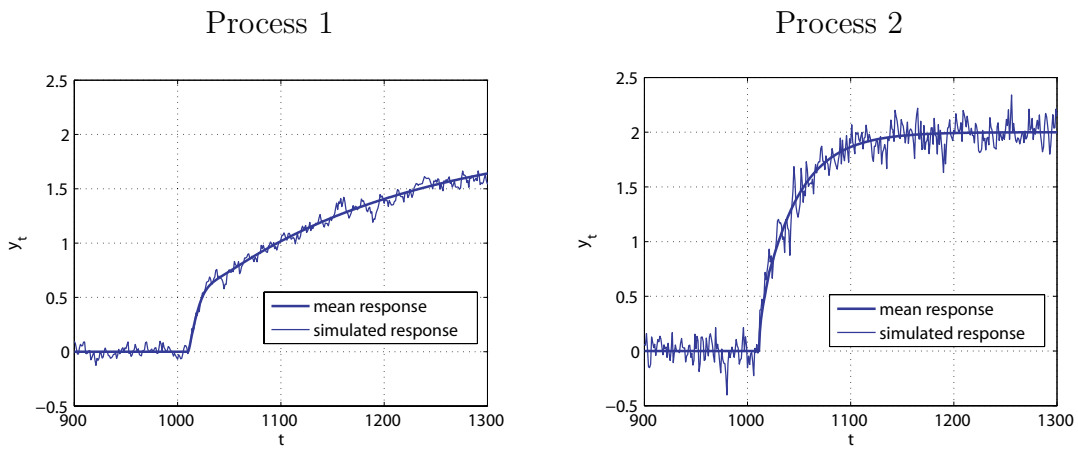


Figure 4: Simulated delay estimation experiments of the two processes in Example 2. For comparison the mean responses of the processes are also shown (thick line).

As a relatively large number of simulations were considered in this example, the full Bayesian approach is not practical due to its increased computational cost. Therefore, only the reduced Bayesian approach was applied. In the SPRT and the CUSUM detection rules, the reference value was set as $b = 3$ for both processes.

In the Bayesian approach, the benefit of having prior knowledge about the change-point r and the time constant γ was investigated. In the prior of r the upper bound on the delay was chosen as $k_0 = 20$ for both processes. A binomial prior was chosen as the informative prior on r for which the probability of success was set at $p_0 = 9/19$ so that its mode is equal to the true

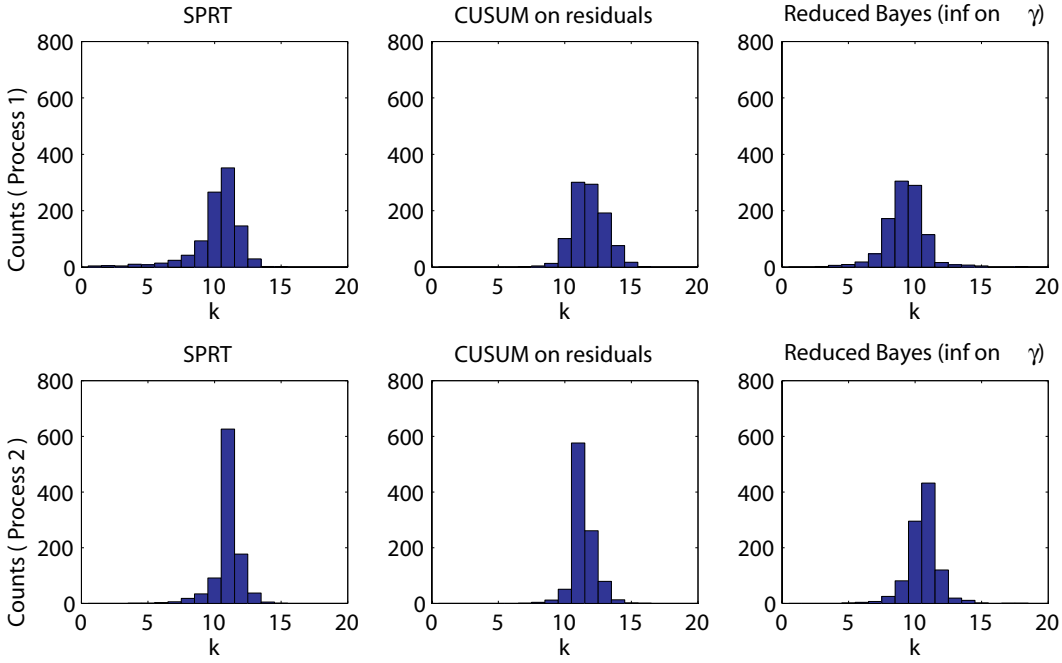


Figure 5: Histograms of the delay estimates of the first process (top panels) and second process (bottom panels) in Example 2 obtained using the SPRT, CUSUM on residuals and the reduced Bayesian (informative prior on γ) approaches. The true delay is $k = 10$ for both processes. Results are based on 1000 simulations.

change point 1010. A log-normal density was chosen as the informative prior on γ for which the location parameter μ_γ was set at the true γ^* values of the two processes. Table 3 gives the informative and noninformative prior scenarios considered.

Scenario	Process 1	Process 2
Non-informative on r and on γ	$\pi(r) = DU(1001, 1020)$ $\pi(\gamma) \propto 1/\gamma$	$\pi(r) = DU(1001, 1020)$ $\pi(\gamma) \propto 1/\gamma$
Informative on r (Non-inf on γ)	$\pi(r) = Bin(20, 9/19, 1000)$	$\pi(r) = Bin(20, 9/19, 1000)$
Informative on γ (Non-inf on r)	$\pi(\gamma) = LN(4.9, 0.05^2)$	$\pi(\gamma) = LN(3.4, 0.3^2)$

Table 3: Scenarios for prior distributions in the reduced Bayesian approach.

Figure 5 shows the histograms of the delay estimates obtained by using the SPRT, CUSUM (on residuals) and the reduced Bayesian (with an informative prior on γ) approaches and Table 4 summarizes the performance statistics of all approaches. This table also gives the performance statistics of the Laguerre approach of Isaksson et al. [17] that we describe in Appendix A.3 (see Figure 8 for the histograms of the delay estimates obtained using the Laguerre approach). As noted in Appendix A.3, the RMSE values we found for the Laguerre method are *lower* (better)

Method	Process 1			Process 2		
	Mean	Var	RMSE	Mean	Var	RMSE
SPRT	10.24	3.30	**1.832**	11.01	1.03	**1.429**
CUSUM (On data)	11.51	2.79	**2.252**	11.28	1.19	**1.680**
CUSUM (On residuals)	11.84	1.55	**2.218**	11.38	0.76	**1.632**
Reduced Bayes (Noninform on r and on γ)	16.74	10.37	**7.472**	14.49	5.17	**5.031**
Reduced Bayes (Inform on r)	10.17	0.79	**0.907**	10.99	0.92	**1.382**
Reduced Bayes (Inform on γ)	9.27	2.24	**1.664**	10.62	1.40	**1.334**
Laguerre	12.29	1.35	**2.570**	11.66	2.02	**2.189**

Table 4: Mean, Variance and Root Mean Square Error (RMSE) of the delay estimates obtained with the SPRT, CUSUM, reduced Bayesian and Laguerre methods from 1000 simulated realizations (Example 2). The prior scenario used in each reduced Bayesian scheme is given in parentheses (see Table 3). The true delay is $k = 10$ for both processes.

than those reported by Isaksson et al. Therefore, these values were used for comparing the proposed approaches to the Laguerre method.

As it can be seen, all approaches (except for the Bayesian approach with an informative prior on r) have a better performance with the second process which has a smaller time constant. This suggests that the performance of all methods improves as the time constant decreases (or as the process dynamics becomes faster). When prior process knowledge is available, the Bayesian approach with an informative prior on the change point r outperforms the alternatives, as expected. When little is known about r but accurate knowledge about the time constant γ is available, the Bayesian approach with an informative prior on γ also provides significantly improved delay estimates than the other methods. By contrast, when no prior process knowledge is available, the SPRT approach results in the best delay estimates. We note how all change point approaches, except for the Bayesian approach with noninformative priors, provide better delay estimates than the Laguerre method. It should be pointed out that the non-parametric CUSUM approach on the original data, which does not require any ARMA modelling, still performs considerably better than the existing Laguerre delay estimation method for both processes.

9 Sensitivity of the SPRT and the Laguerre delay estimation methods

As discussed in Appendix A.1, the reference value b of the SPRT method must be selected based on the dynamics of the process. However, in practice the appropriate reference value for a particular process dynamics may not be easily obtainable.

In this section we study the sensitivity of the SPRT delay estimation method with respect

to the choice of the reference value b and compare it to the sensitivity of the Laguerre method with respect to the choice of its frequency parameter ω. The sensitivity analysis consists of estimating the delay of a given process under a given step shift size by employing SPRT with various values of b and by employing the Laguerre method with various values of ω.

Consider the process model given by equation (28) controlled by the PI controller and the delay estimation experiment with the step shift sizes $\delta = 3\sigma_y$ and $\delta = 6\sigma_y$ (i.e., the signal-to-noise ratios $a = 3$ and $a = 6$). Recall that the true delay of the process is $k = 3$. To study the sensitivity of the methods, the experiment is simulated 100 times and during each experiment the delay is estimated by applying the following:

(i) The SPRT method and using the range of reference values $0.875 \leq b \leq 2$ for $a = 3$ and using $1.75 \leq b \leq 4$ for $a = 6$

(ii) The Laguerre method and using the range of frequency values $1 \leq \omega \leq 2.8$ for $a = 3$ and using $0.5 \leq \omega \leq 1.4$ for $a = 6$.

Under each signal-to-noise ratio, the ranges of the b and ω values were determined so that they cover the minimum of the delay estimate root mean square error (RMSE) value. As it can be seen, in order to achieve the minimum RMSE under a smaller signal-to-noise ratio, in SPRT a smaller reference value has to be used and in the Laguerre method a larger frequency has to be used.

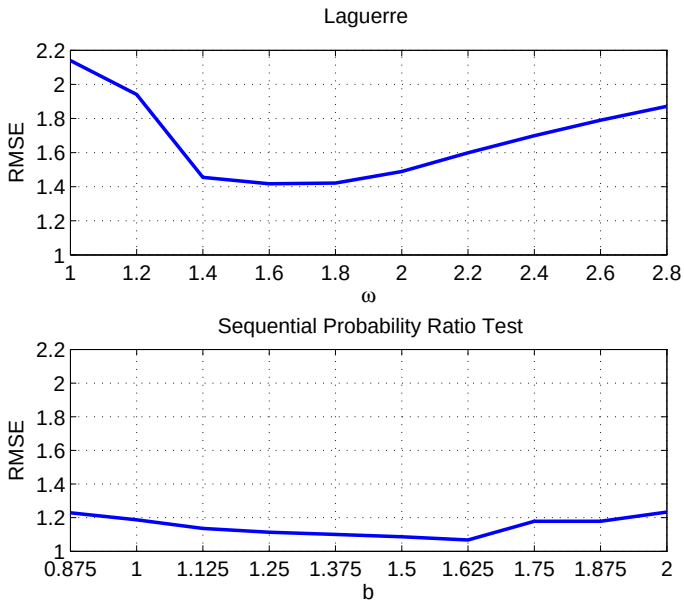
Figure 6 shows the RMSE of the delay estimates obtained with SPRT and the Laguerre method and under increasing b and ω values. As it can be seen, under both shift sizes the SPRT method gives a better RMSE performance than the Laguerre approach regardless of the b and ω parameters used. This indicates that the SPRT method is less sensitive to the choice of its tuning parameter than is the Laguerre method.

10 Conclusions

This paper presented three new closed-loop time-delay estimation methodologies that are based on Bayesian, SPRT and CUSUM change-point detection methods. Performance comparisons were made against the Laguerre method in [17]. In the examples, processes that have first-order step responses were considered. It was shown that the change point approaches work for both first and second order responses, with the latter being more favorable due to the possibility of overshoot.

The CUSUM approach yielded more precise delay estimates when applied on residuals than when applied on original data. The SPRT and the CUSUM (on residuals) approaches provided comparable quality delay estimates. It was demonstrated that even when little is known about the delay a Bayesian approach can be used to incorporate prior information about a different process parameter (such as the time constant) to obtain significantly better

(a)



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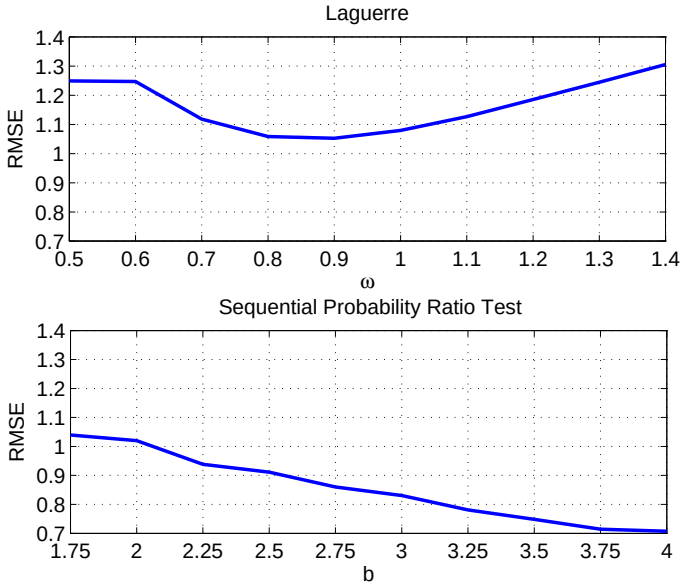

Figure 6: RMSE of the delay estimates obtained with SPRT and Laguerre methods under increasing b and ω values. (a) Small step shift ($\delta = 3\sigma_y$). (b) Large step shift ($\delta = 6\sigma_y$). Each RMSE is computed from 100 simulations of the delay estimation experiment.

delay estimates than the other methods. When no prior process knowledge exists, however, the SPRT approach yielded the best estimates. It was also shown that the proposed approaches give better performance than the existing Laguerre method [17].

Except for the CUSUM delay estimation applied to the original data, all proposed methods were compared against the Laguerre method assuming the correct model orders of the closed-loop ARMA description of the process were known exactly. Although not a strict assumption, since these orders can be estimated from regular closed-loop operating data using standard time-series methods, it should be pointed out that the CUSUM on the data (which requires no ARMA model be used) still overperformed the Laguerre method in our comparisons (note we obtained better performance of the Laguerre method than that reported in [17], see Appendix A.3).

Finally, the sensitivity analysis of the SPRT and the Laguerre delay estimation methods showed that the SPRT method is less sensitive to the selection of its tuning parameter than the Laguerre method. Therefore the SPRT method is easier to tune with respect to the true dynamical response of the process and performs better overall.

A Appendix

A.1 Selection of the reference values of the SPRT and the CUSUM schemes.

Performance of a Statistical detection scheme is usually assessed based on its run length (RL) characteristics. The RL is defined as the number of samples required for the scheme to signal a change. It is desirable to have an "in-control" RL (the RL when no change occurs in the process) as high as possible and an "out-of-control" RL (the RL when a change occurs in the process) as small as possible ([11], pp. 12).

In order to achieve a good out-of-control RL performance in a delay estimation test, the reference value b of the SPRT and CUSUM methods must be selected based on both the actual shift size a and the dynamical response of the process. With respect to actual shift size we need to have that $b < a$. Figure 7 illustrates the performance of a detection rule in two processes that have different dynamical behavior. In this case b_1 and b_2 are the reference values used in the faster and the slower dynamics processes, respectively. As it can be seen, in order to obtain the same out-of-control RL in both processes, one must use a smaller reference value in the process with slower dynamics, that is, one must have that $b_2 < b_1$.

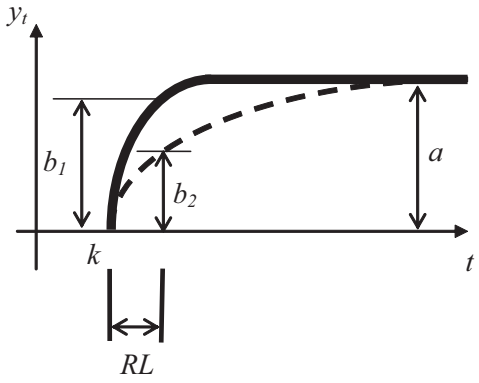

Figure 7: Effect of the selection of the reference value b on the out-of-control run length (RL) performance of the SPRT and the CUSUM schemes. **Solid line:** the step response of a process with faster dynamics. **Dashed line:** the step response of a process with slower dynamics. Here, a is the actual shift size and $b < a$.

A.2 Estimation of the parameters required in the change-point methods.

Define $\boldsymbol{Y}^1 = \{y_t\}_{t=1}^{s}$ as the vector of output data before making the set-point change and $\boldsymbol{Y}^1 = \{y_t\}_{t=s+k_0}^{n}$ as the vector of output data after the effect of the change is fully realized (recall that k_0 is the upper bound on the delay). Similarly, define $\boldsymbol{E}^1 = \{e_t\}_{t=1}^{s}$ and $\boldsymbol{E}^2 = \{e_t\}_{t=s+k_0}^{n}$ as the vectors of the residuals (before and after the set-point change) of an ARMA model fitted to the output data.

In the reduced Bayesian and SPRT approaches, $\mu_1, \boldsymbol{\phi}, \boldsymbol{\theta}$ and σ^2 are estimated from $\boldsymbol{Y}^1$, and μ_2 is estimated from $\boldsymbol{Y}^2$. In the CUSUM on original data μ_1 and σ_y^2 are estimated from $\boldsymbol{Y}^1$ and μ_2 is estimated from $\boldsymbol{Y}^2$. In the CUSUM on residuals $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ are estimated from $\boldsymbol{Y}^1$, and (μ_{1e}, σ_e) and μ_{2e} are estimated from $\boldsymbol{E}^1$ and $\boldsymbol{E}^2$. The *armax* function in MATLAB was used to estimate the ARMA parameters.

A.3 Delay Estimation Using Laguerre Transfer Function Models

This appendix reviews the delay estimation method employed by Isaksson et. al. [17]. The Laguerre transfer function model of a process can be written as

$$y_t = \sum_{i=1}^{n_c} c_i L_i(z^{-1}) u_t + v_t \tag{33}$$

where

$$L_i(z^{-1}) = \frac{\sqrt{1-\alpha^2}z^{-1}}{1-\alpha z^{-1}} \left(\frac{z^{-1}-\alpha}{1-\alpha z^{-1}} \right)^{i-1}, \quad i = 1, .., n_c$$

where α is the parameter and n_c is the order of the transfer function model. In [17] the values $n_c = 10$ and $\alpha = 0.8$ are recommended.

The coefficients c_i of model (33) can be estimated using ordinary least squares by fitting to the input-output data the following linear regression model

$$y_t = c_1 u_{1,t} + c_2 u_{2,t} + ... + c_{n_c} u_{n_c,t} + v_t \tag{34}$$

where $u_{i,t} = L_i(z^{-1})u_t \;\; (i = 1, 2, ..., n_c)$ and it is obtained by filtering the original input signal u_t with the i-th Laguerre filter L_i. Matlab's *filter* command is used for this purpose.

In order to estimate the delay, the estimated Laguerre transfer function is first represented in a rational transfer function form:

$$y_t = G(z^{-1})u_t + v_t \tag{35}$$

where $G(z^{-1}) = \sum_i c_i L_i(z^{-1})$. Then, the rational transfer function is factorized into a minimum phase and an all phase part as $G(z^{-1}) = G_{mp}(z^{-1})G_{ap}(z^{-1})$. The minimum phase part contains the unstable roots (the roots that are inside the unit circle) of the numerator polynomial of $G(z^{-1})$. The all phase part G_{ap} is written in the frequency domain by the substitution $z^{-1} = e^{-i\omega T_s}$ where T_s is the sampling interval (in seconds) and ω is the frequency parameter (in radians/second). Finally, the time delay (in seconds) is estimated as

$$\hat{T}_d = \lim_{\omega \to 0} \left(-\frac{\varphi(\omega)}{\omega} \right) \tag{36}$$

where $\varphi(\omega) = \arg\{G_{ap}(e^{-i\omega T_s})\}$ is the phase angle (in radians) of $G_{ap}(e^{-i\omega T_s})$ and it can be computed using the *angle* command in Matlab. In expression (36) the quantity within the large parentheses gives the time delay at a given frequency. Therefore, in order to approximate the pure-delay (i.e. low frequency) component of the transfer function, the limit of this quantity is taken as the frequency tends to zero. From (36) the delay (in time periods) is estimated as

$$\hat{k} = 1 + \frac{\hat{T}_d}{T_s}.$$

It is noted in Isaksson et. al. [17] that in order to approximate the limit, a sufficiently small ω must be specified by the user based on the noise level of the data. It was reported that the value $\omega = 10^{-4}$ provided adequate approximation for typical industrial data.

As an illustration, we estimate the delay of the two processes considered in [17] using the Laguerre delay estimation method and compare the root mean squared error (RMSE) of the delay estimates from 1000 simulations to that reported in [17]. We refer to equations (31) and (32) for the descriptions of these processes. The delay estimation experiment is of length $n = 2000$ and a step shift of size $\delta = 2$ is introduced to the process at time $s = 1000$.

In each simulation a Laguerre transfer function model with parameters $n_c = 10$ and $\alpha = 0.8$ was fitted to the step response data. In order to estimate the delay, the phase angle of the estimated frequency domain transfer function model was evaluated at $\omega = 10^{-4}$ for the first

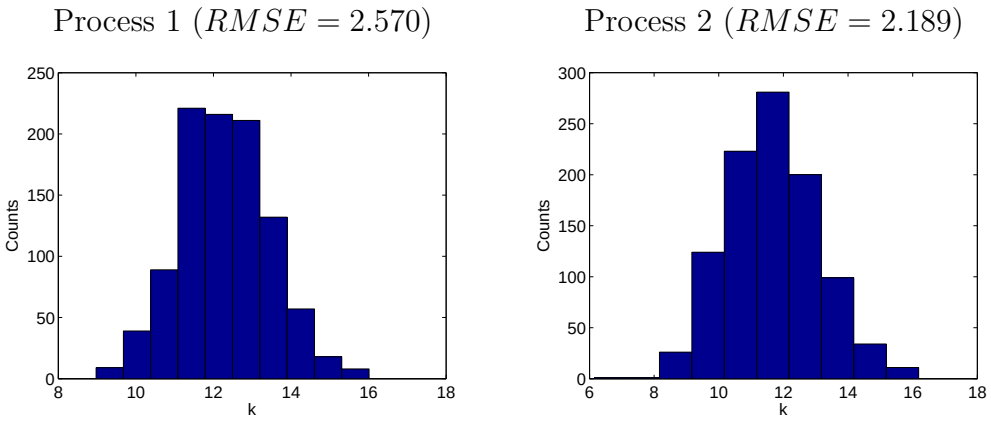

Figure 8: Histograms and RMSE values of the delay estimates of the two processes considered in [17]. Estimates are obtained with the Laguerre method from 1000 simulations. True delay is 10.

process and $\omega = 0.08$ for the second process. The histograms and the RMSE values of the delay estimates obtained from the simulations are given in Figure 8. The RMSE of the estimates computed with the Laguerre approach (using the same number of simulations) is reported by [17] as 3.529 and 2.412 for the first and second processes, respectively. As it can be seen, our results indicate that the Laguerre approach has a better RMSE performance than that originally reported in [17].

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