On-line Process Adjustment for Asymmetric Cost functions

B.M. Colosimo[1], R. Pan[2], E. del Castillo[2*]

[1] Dipartimento di Meccanica - Politecnico di Milano
P.za Leonardo da Vinci 32, 20133 Milano, Italy

[2]Department of Industrial & Manufacturing Engineering
The Pennsylvania State University, University Park, PA 16802

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Abstract

This paper presents a feedback control rule for the machine start-up adjustment problem when the cost function of the machining process is not symmetric around its target. In particular, the presence of a bias term in the control rule permits the process quality characteristic to converge to a steady-state target from the lower cost side, thus reducing the process quality losses incurred during the transient phase of adjustment. A machining application is used to demonstrate the savings generated by the biased linear feedback adjustment rule compared to an adjustment rule due to Grubbs (1954, 1983) and to an integral (or EWMA) controller. The performance of the different adjustment schemes is studied from a small-sample point of view, showing that the advantage of the proposed rule is significant especially for expensive parts which are usually produced in small lots. In this paper, two asymmetric cost functions – constant and quadratic – are considered. Optimal biased control rules for both cost functions are derived.

Keywords: process control, feedback adjustment, stochastic approximation, one-sided convergence

1 Introduction

After an imprecise setup or maintenance operation, a machine can produce a systematic process error which will show on the quality characteristic of the machined items. Adjustments are necessary for eliminating such error if there are some controllable variables that can be manipulated on the machine. However, the start-up error is unobservable directly due to the inherent randomness of both the machining and measurement processes. Therefore, a sequence of adjustments that utilizes the process information obtained on-line is useful for eventually removing the start-up error. Grubbs (1954, 1983) proposed such an adjustment rule which has been more recently discussed by Trietsch (1998) and del Castillo and Pan (2001). The latter reference shows the connections between Grubbs rule and stochastic approximation techniques.

*corresponding author

Former research on the start-up adjustment procedure only dealt with the case of symmetric cost functions. It is well-known that in industrial practice asymmetric cost functions can be more appropriate, since the cost of oversized and undersized quality characteristics are often different, like, for instance, in hole-finishing or milling operations. The impact of asymmetric cost functions has been studied from several perspectives. Wu and Tang (1998) and Maghsoodloo and Li (2000) have considered tolerance design with asymmetric cost functions, while Moorhead and Wu (1998) have analyzed the effect of this type of cost function on parameter design. Ladany (1995) presented a solution to the problem of setting the optimal target of a production process *prior* to starting the process under a constant asymmetric cost function. Harris (1992) discussed the design of minimum-variance controllers with asymmetric cost functions for a process characterized by a linear dynamic model and ARIMA (AutoRegressive Integrated Moving Average) noise. Despite of the generality of this model, a possible process start-up error has not been included into consideration.

When start-up errors exist under an asymmetric cost function, it is intuitive to have the value of the quality characteristic converge to the optimal setting from the lower cost side. This is related to certain stochastic approximation techniques in which a bias term is added to allow for one-side convergence, as discussed by Anbar (1977) and Krasulina (1998). However, these approaches are oriented to asymptotic or long-term performance, and the conditions they impose on the control rule parameters are too complicated for practical engineering application. Since short-run production processes have become more common with the advent of modern manufacturing environments, small sample properties of sequential adjustment procedures need to be studied.

In this paper, we propose a generic framework for the start-up adjustment problem for asymmetric cost functions and focus on its small sample performance. First, two asymmetric cost functions representing two different cost models used in manufacturing are presented. We include a bias term into a general linear control rule. The optimal value of this bias term in the sense of minimizing the expected manufacturing cost at each time step is then derived. The proposed procedure is compared with other adjustment methods in the literature by evaluating and comparing their short-run costs. Finally, a real manufacturing process is used to demonstrate the practical application of our adjustment procedure for asymmetric cost functions.

2 Process and cost models

Suppose the quality characteristic Y_n of each machined part is measured with reference to a nominal value, which is assumed, without loss of generality, to be equal to 0. After the start-up, the process is supposed to be off-target by d units, i.e., $Y_1 = d + \varepsilon_1$, where ε_1 models both the inherent production variability and the error of measurement. After the first quality characteristic is measured the value of the control parameter U_1, which is assumed to have an immediate effect on the process output, is set, thus inducing a change in the next quality characteristic: $Y_2 = d + U_1 + \varepsilon_2$. The procedure is thus iterated and the general expression for the quality characteristic at the n^{th} step, Y_n, is given by:

$$Y_n = d + U_{n-1} + \varepsilon_n \tag{1}$$

where:

- $n = 1, ..., N$ denotes a discrete time index or part number;
- U_{n-1} is the value of the controllable variable at the $n-1^{th}$ step of the adjustment procedure, with $U_0 = 0$;
- d is the initial unknown offset (a constant);
- $\{\varepsilon_n\}$ represents normally distributed white noise: $\varepsilon_n \sim N(0, \sigma_\varepsilon^2)$, thus the errors are i.i.d. random variables.

To evaluate the costs associated with the control procedure, two cost models often adopted in industrial practice are considered. In the first case, costs are assumed to arise only when the part processed is non-conforming, i.e., when the quality characteristic is out of the Specification Limits. In particular, it will be assumed that the violation of the Lower or the Upper Specification Limit could lead to different costs. For example, consider the case of a quality characteristic related with a dimension obtained after a finishing operation. In such operation, the costs associated with oversized and undersized items, which are mainly determined by either scrapping or re-working, are almost always different.

Therefore, two constants, c_1^c and c_2^c, are used to represent the costs associated with the violation of the LSL and USL, respectively. The superscript c indicates the *constant* cost model, given by:

$$C_n^c = \begin{cases} c_1^c & \text{if } Y_n < LSL \\ 0 & \text{if } LSL \leq Y_n \leq USL \\ c_2^c & \text{if } Y_n > USL \end{cases} \tag{2}$$

(see Figure 1).

Another asymmetric cost model considered is based on a piecewise quadratic cost function. In this case, the cost function can be more properly considered as a penalty function, in which the loss is assumed to be proportional to the square of the distance of the quality characteristic from its nominal value. The asymmetry in the cost function is modeled through two constants, c_1^q and c_2^q, where the superscript q indicates the *quadratic* cost model, given by

$$C_n^q = \begin{cases} c_1^q Y_n^2 & \text{if } Y_n < 0 \\ c_2^q Y_n^2 & \text{if } Y_n \geq 0\ . \end{cases} \tag{3}$$

The value of the constants c_1^q and c_2^q can be computed with reference to the Specification Limits as suggested by Wu and Tang (1998). The distance between the nominal value and the LSL or USL

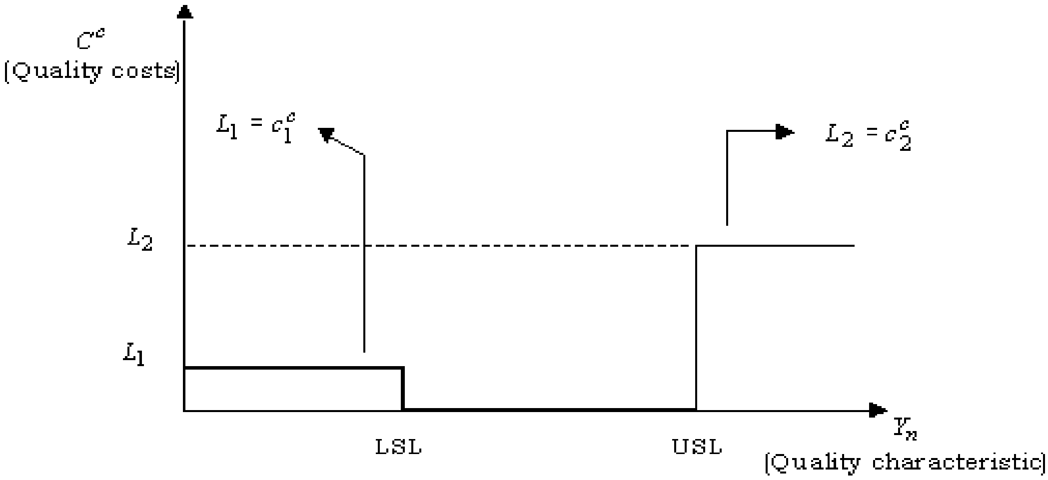

Figure 1: The asymmetric constant cost function with different costs when the quality characteristic is below LSL or above USL.

is denoted by Δ, and the cost corresponding to a quality characteristic equal to LSL or USL is L_1 or L_2, respectively. The constants, c_1^q and c_2^q, are given by:

$$c_1^q = \frac{L_1}{\Delta^2} \quad \text{and} \quad c_2^q = \frac{L_2}{\Delta^2} \tag{4}$$

(see Figure 2). The correspondence between the coefficients adopted with the constant and the quadratic cost models can be found from (4) by letting $L_1 = c_1^c$ and $L_2 = c_2^c$. The traditional symmetric cost models are therefore special cases of the above models, i.e., $c_1^c = c_2^c$ and $c_1^q = c_1^q$.

Since most of the recently developed devices for on-line inspection and measurement can transmit the data acquired to the controller of the machine, the assumption of an automatic feedback procedure is realistic. In this scenario, the cost of the adjustments can be neglected and therefore has not been considered in the following analysis.

The asymmetry in the cost function implies two issues that have to be considered in designing the adjustment rule. The first is related to the long-term or steady-state target $T^\bullet$ that has to be entered on the machine at start-up, where the superscript $^\bullet$ is replaced by either c or q to indicate either constant or quadratic cost function. The problem of determining this value, referred to in the literature as the *optimum target point*, has been addressed for asymmetric cost functions in manufacturing by Ladany (1995) and Wu and Tang (1998).

The second issue is related to the way in which, starting from an initial offset, the quality characteristic should converge to the target as determined by the adjustment procedure. Both of these issues are considered in the remainder of the paper. In particular, the steady-state target $T^\bullet$ will be derived by minimizing the long term expected costs, and the adjustment rule will be determined by considering all the costs associated with the transient period, evaluating the Average

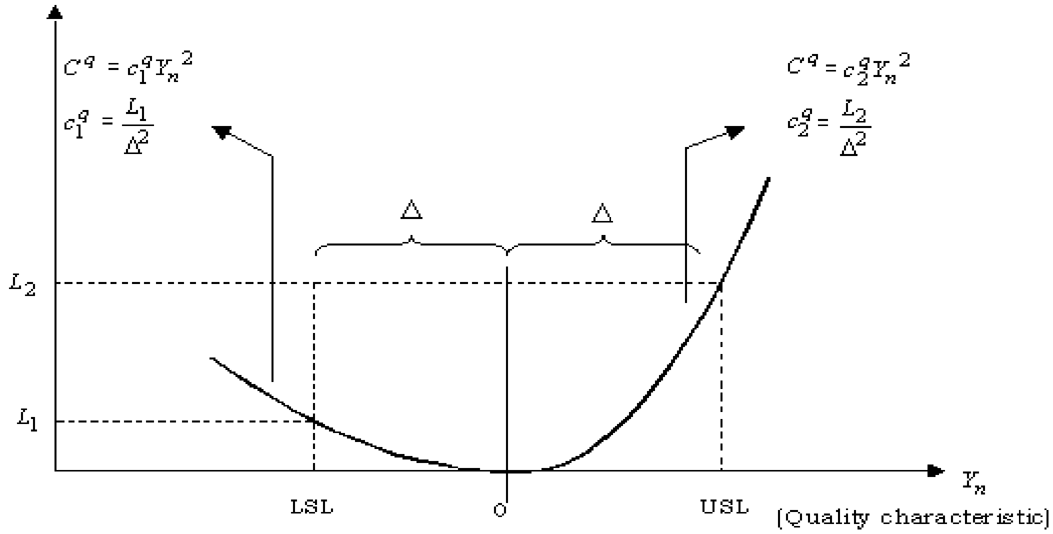

Figure 2: The asymmetric quadratic cost function with different costs when the quality characteristic is below LSL or above USL.

Integrated Expected Cost (AIEC) performance index:

$$AIEC^{\bullet} = \frac{1}{N} \sum_{n=1}^{N} E(C_n^{\bullet}) \tag{5}$$

where $E(C_n^{\bullet})$ indicates the expected value of the costs at the n^{th} step of the adjustment procedure.

3 The Biased feedback adjustment rule

Since a control variable is available for removing a possible start-up error of a process, it is necessary to design a feedback adjustment rule to manipulate this variable. A common feedback linear adjustment rule is one of the form:

$$U_n = U_{n-1} - k_n(Y_n - T^{\bullet}) \ . \tag{6}$$

That is, the adjustments $U_n - U_{n-1}$ are proportional to the latest measured deviation of the quality characteristic Y_n from the steady-state target $T^{\bullet}$. Del Castillo and Pan (2001) showed that, depending on the selection of the sequence $\{k_n\}$, this form of feedback adjustment results in Grubbs' rule (Grubbs, 1954, 1983, which in turn is a direct application of Robbins and Monro's, 1951, Stochastic Approximation techniques), the EWMA or integral controller (Box and Luceño, 1995), the Kalman filter (Kalman, 1960), and an approach based on Recursive Least Squares. The performance of all the rules mentioned above has been studied in the literature only with respect to symmetric cost functions.

Since the asymmetry in the cost model induces different losses depending on the side from which the quality characteristic approaches the steady-state target, the performance of the linear adjustment rule could be enhanced by introducing a bias term in (6). Anbar (1977) proposed a biased stochastic approximation procedure, further studied by Krasulina (1998), for the problem of one-side convergence. In this model, a bias term b_n is introduced into the adjustment rule, i.e.,

$$U_n = U_{n-1} - k_n(Y_n - T^{\bullet} + b_n) \ . \tag{7}$$

Using the law of the repeated logarithm, Anbar demonstrated the convergence of Y_n as $n \to \infty$ when b_n converges to zero in $n^{\frac{1}{2}}(log(log\ n))^{-\frac{1}{2}}$.

Equation (7) is the adjustment rule we will consider in what follows; however, the conditions of process variables outlined in Anbar (1977) do not give insight on the selection of the sequence $\{b_n\}$ with reference to a specific asymmetric cost function. The adjustment procedure proposed in this paper is instead oriented to derive a sequence of bias coefficients $\{b_n\}$ that minimize the costs incurred during the transient phase of convergence of the quality characteristic to its steady-state target. In order to preserve its easiness of use, the bias sequence $\{b_n\}$ should be able to be computed off-line even when the process measurements are not available. This condition assures the control rule to be applicable to any manufacturing process, independently from the time units characterizing its dynamics.

By recursively substituting (7) in (1), the general expression of the quality characteristic at the n^{th} step of the procedure is given by:

$$Y_n = \prod_{i=1}^{n-1}(1-k_i)d - \sum_{i=1}^{n-1}\left[k_i(\varepsilon_i + b_i)\prod_{j=i+1}^{n-1}(1-k_j)\right] + T^{\bullet}\sum_{i=1}^{n-1}\left[k_i\prod_{j=i+1}^{n-1}(1-k_j)\right] + \varepsilon_n \tag{8}$$

where:

$$\prod_{j=n}^{n-1}(1-k_j) = 1 \ .$$

Since process errors are normally distributed, the quality characteristic Y_n at each step of the procedure is also normally distributed, i.e., $Y_n \sim N(\mu_n, \sigma_n^2)$, with mean and variance equal to:

$$\mu_n = \prod_{i=1}^{n-1}(1-k_i)d - \sum_{i=1}^{n-1}\left[k_i b_i\prod_{j=i+1}^{n-1}(1-k_j)\right] + T^{\bullet}\sum_{i=1}^{n-1}\left[k_i\prod_{j=i+1}^{n-1}(1-k_j)\right] \tag{9}$$

$$\sigma_n^2 = \sigma_\varepsilon^2\left[1 + \sum_{i=1}^{n-1}k_i^2\prod_{j=i+1}^{n-1}(1-k_j)^2\right] \ . \tag{10}$$

As it can be observed, the sequence of bias terms $\{b_i\}$ affects only the mean value μ_n of the quality characteristic. Therefore, for a given selection of $\{k_i\}$, the bias terms $\{b_i\}$ can be determined by equating the right hand side of expression (9) to the *optimal mean* at the n^{th} step $m_n^{\bullet}$, i.e., the n^{th} component of the vector $\mathbf{m}^{\bullet} = \{m_n^{\bullet}, n = 1, ..., N\}$ that minimizes the $AIEC^{\bullet}$ given by (5). The computation of $\mathbf{m}^{\bullet}$ will be addressed in the next section.

Although the approaches in Anbar (1977) and Krasulina (1998) utilize the harmonic sequence for $\{k_n\}$, i.e., $k_n = 1, 1/2, 1/3, ...$, it is in principle possible to consider a different sequence, while maintaining the form of the controller given by (7). For example, besides considering the harmonic sequence (Grubbs' approach), a constant sequence (the EWMA or integral control approach) can be considered instead.

In the case when $k_i = 1/i$, $i = 1, 2, ..., n-1$ (a harmonic series), the value of the mean and the variance of the quality characteristic at each step are given by:

$$\mu_n = T^\bullet - \frac{1}{n-1}\sum_{i=1}^{n-1} b_i \tag{11}$$

$$\sigma_n^2 = \sigma_\varepsilon^2 \left(\frac{n}{n-1}\right) . \tag{12}$$

If k_i is instead set equal to a constant λ, as in the EWMA approach, the resulting mean and variance are:

$$\mu_n = T^\bullet + (1-\lambda)^{n-1}(d - T^\bullet) - \lambda(1-\lambda)^{n-1}\sum_{i=1}^{n-1}\frac{b_i}{(1-\lambda)^i} \tag{13}$$

$$\sigma_n^2 = \sigma_\varepsilon^2 \left[\frac{2 - \lambda(1-\lambda)^{2(n-1)}}{2-\lambda}\right] . \tag{14}$$

It is noticed that in Equation (11) the value of μ_n does not depend on the initial unknown offset d, thus an off-line computation of b_n is possible. For the biased EWMA approach, μ_n (eq. 13) is a function of the unknown offset d, so the sequence of biased coefficients $\{b_n\}$ can not be computed off-line. Therefore, we will only consider the biased harmonic adjustment rule in what follows. From Equation (11), the general expression for b_n can be obtained by equating the mean of the response to the optimal mean at the n^{th} and the $n+1^{th}$ steps, i.e.,

$$-\frac{1}{n-1}\sum_{i=1}^{n-1} b_i + T^\bullet = m_n^\bullet$$

and

$$-\frac{1}{n}(\sum_{i=1}^{n-1} b_i + b_n) + T^\bullet = m_{n+1}^\bullet ,$$

from where the general expression for the bias term b_n is given by

$$b_n = n(T^\bullet - m_{n+1}^\bullet) - (n-1)(T^\bullet - m_n^\bullet) . \tag{15}$$

4 The optimal target and the sequence of bias terms

To complete the adjustment rule, the optimal steady-state target $T^\bullet$ and the sequence of bias terms $\{b_n\}$ have to be specified. As previously mentioned, the first value represents the optimal mean

$m_n^\bullet$ as $n \to \infty$, while the sequence of bias terms can be instead computed by using (15), once the vector of optimal means $\mathbf{m}^\bullet$ is known. To compute this vector, the minimization problem that has to be solved can be stated as:

$$min_{\boldsymbol{\mu}} \; AIEC^\bullet \tag{16}$$

where $\boldsymbol{\mu} = \{\mu_n, n = 1, ..., N\}$ is the vector composed by the means of the response at each step of the procedure, and $AIEC^\bullet$ is the performance index given by equation (5). As showed in Appendix A.1, when the linear control rule (7) is in use, the optimization in (16) is equivalent to the following set of minimization problems:

$$min_{\mu_n} \; E(C_n^\bullet) \; , \; n = 1, 2, ..., N \; . \tag{17}$$

Problems in (17) will be solved for the two types of cost functions studied.

Consider first the constant asymmetric cost function. The expected cost at time n is given by:

$$\begin{aligned} E(C_n^c) &= c_1^c \int_{-\infty}^{LSL} f_N(y_n; \mu_n, \sigma_n^2) dy_n + c_2^c \int_{USL}^{\infty} f_N(y_n; \mu_n, \sigma_n^2) dy_n \\ &= c_1^c \Phi\left(\frac{LSL - \mu_n}{\sigma_n}\right) + c_2^c \left[1 - \Phi\left(\frac{USL - \mu_n}{\sigma_n}\right)\right] \end{aligned} \tag{18}$$

where $f_N(\cdot)$ is the normal density function and $\Phi(\cdot)$ is the standard normal distribution function. The minimum of this function with respect to μ_n can be derived by computing the first and second order derivatives of $E(C_n^c)$. As reported in Appendix A.2, the optimal mean m_n^c, obtained by equating the first derivative of $E(C_n^c)$ to zero, is given by

$$m_n^c = \frac{\sigma_n^2 \ln(\frac{c_1^c}{c_2^c})}{(USL - LSL)} + \frac{1}{2}(USL + LSL). \tag{19}$$

As a special case, when the cost function is symmetric, i.e. $c_1^c = c_2^c$, the result obtained is $m_n^c = (USL + LSL)/2$, which is equal to 0 when USL and LSL are symmetric around the nominal value. Since the second derivative with respect to μ_n (equation (36) in Appendix A.2) is always greater than zero when the condition $LSL < \mu_n < USL$ is satisfied, the value of m_n^c obtained is the minimum for the expected cost $E(C_n^c)$.

The steady-state target T^c can be derived as a particular case of the general expression (19) by considering the limit, as $n \to \infty$, of σ_n^2 given by (12). Since this limit is equal to σ_ε^2, we get

$$T^c = \frac{\sigma_\varepsilon^2 \ln(\frac{c_1^c}{c_2^c})}{(USL - LSL)} + \frac{1}{2}(USL + LSL) \; . \tag{20}$$

Substituting (20) and (19) into the expression of the bias term, given by (15), the values of the bias terms b_n for the asymmetric constant cost function can be directly computed. In this case, all b_n's except the first one equal to zero, i.e.,

$$b_n = \begin{cases} -\frac{\ln(\frac{c_1^c}{c_2^c})\sigma_\varepsilon^2}{(USL-LSL)} & \text{if n=1} \\ 0 & \text{if n=2,...,N .} \end{cases} \tag{21}$$

Although the feedback adjustment procedure has a non-zero bias b_n only at the first step, b_1 affects the following adjustments through the U_{n-1} term in the expression of the controller (7).

Consider now the quadratic asymmetric cost function. The expected cost at the n^{th} step of the procedure is given by

$$E(C_n^q) = c_1^q \int_{-\infty}^{0} y_n^2 f_N(y_n; \mu_n, \sigma_n^2) dy + c_2^q \int_{0}^{\infty} y_n^2 f_N(y_n; \mu_n, \sigma_n^2) dy_n.$$

By solving the two integrals (as reported in Appendix A.3), the following expression for the expected value of the cost is obtained:

$$E(C_n^q) = c_2^q(\mu_n^2 + \sigma_n^2) + (c_2^q - c_1^q)\left[\sigma_n \mu_n \phi\left(\frac{\mu_n}{\sigma_n}\right) - (\mu_n^2 + \sigma_n^2)\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] . \tag{22}$$

Computing the first derivative with respect to μ_n and equating it to zero (as reported in Appendix A.4), the optimal mean m_n^q is determined by the following equation:

$$2c_2^q m_n^q + 2(c_2^q - c_1^q)\left[\sigma_n \phi\left(\frac{m_n^q}{\sigma_n}\right) - m_n^q \Phi\left(-\frac{m_n^q}{\sigma_n}\right)\right] = 0 \tag{23}$$

where $\phi(\cdot)$ is the standard normal density function and $\Phi(\cdot)$ is the standard normal distribution function. Although there is no closed form expression for m_n^q, it can be computed numerically off-line, since all the quantities in expression (23) do not depend on the actual observations of the quality characteristic. Similarly as the constant cost function, if the quadratic cost function is symmetric, i.e., $c_1^q = c_2^q$, the optimal mean m_n^q is zero for $n = 1, 2, ..., N$.

The second derivative of $E(C_n^q)$ with respect to μ_n is always positive (as shown in equation (45) in Appendix A.3), so m_n^q given by equation (23) determines a minimum of the expected cost. Again, the steady-state target T^q can be computed as a special case by considering $lim_{n\to\infty}\sigma_n = \sigma_\varepsilon$, in equation (23), so T^q is the solution of

$$2c_2^q T^q + 2(c_2^q - c_1^q)\left[\sigma_\varepsilon \phi\left(\frac{T^q}{\sigma_\varepsilon}\right) - T^q \Phi\left(-\frac{T^q}{\sigma_\varepsilon}\right)\right] = 0 . \tag{24}$$

Therefore, in the case of the quadratic cost model, the feedback adjustment rule can be obtained by evaluating numerically the optimal means m_n^q that satisfy equation (23) for $n = 1, 2, ..., N$, and the optimal target T^q can be obtained from equation (24). Substituting these values in equation (15), we obtain the sequence of bias coefficients $\{b_n\}$.

In summary, the biased linear adjustment procedure for constant and quadratic cost functions are as follows:

Solution to the Asymmetric Constant Cost Model
Given: c_1^c, c_2^c, USL, LSL, σ_ε, N.

1. Compute the steady-state target T^c using (20);

2. Compute the bias coefficient b_1 using (21);

3. Adjust the control variable on-line according to the following equation:

$$U_n = \begin{cases} -[Y_1 - T^c + b_1] & \text{if } n = 1 \\ U_{n-1} - \frac{1}{n}[Y_n - T^c] & \text{if } n = 2, ..., N \ . \end{cases} \tag{25}$$

Solution to the Asymmetric Quadratic Cost Model
Given: c_1^q, c_2^q, σ_ε, N.

1. Compute the steady-state target T^q by solving numerically equation (24);

2. Find the sequence of bias terms $\{b_n\}$ for $n = 1, ..., N$:

   - Compute the optimal mean m_n^q by solving numerically equation (23) where $\sigma_n = \sigma_\varepsilon\sqrt{\frac{n}{n-1}}$;
   - Substitute T_n^q and m_n^q into (15) to obtain b_n;

3. Adopt the biased linear adjustment rule for on-line process adjustment:

$$U_n = U_{n-1} - \frac{1}{n}(Y_n - T^q + b_n) \ .$$

5 An application to a real machining process

In this section, the biased linear adjustment procedure for start-up errors will be applied to a real machining problem. The performance of the biased rule will be compared with that of Grubbs' rule and with the EWMA (integral) controller. The latter two procedures follow the adjustment rules of the form (6) where k_n is equal to $1/n$ for Grubbs' rule and equal to a constant λ for the EWMA controller.

A hole-finishing operation is performed on a pre-existing hole in a raw aluminum part made by pressure casting. The Specification Limits on the final hole diameter are at 57.000 $\pm 0.030mm$. After the execution of the operation, the diameter of the hole (D) is measured in an automatic inspection station constituted by a probe that acquires the diameter while the workpiece rotates 360 degrees around the axis of the hole. The mean diameter is computed and recorded. Due to the materials machined and the tools used (polycrystalline inserts), the tool wear can be neglected and no trend is present in the data collected. We let the quality characteristic of this process be the difference between measurement D_n and the nominal value of the hole diameter, i.e., $Y_n = D_n - 57000$, in microns. The standard deviation of the process σ_ε is estimated through $\sqrt{MSE}$ (the square root

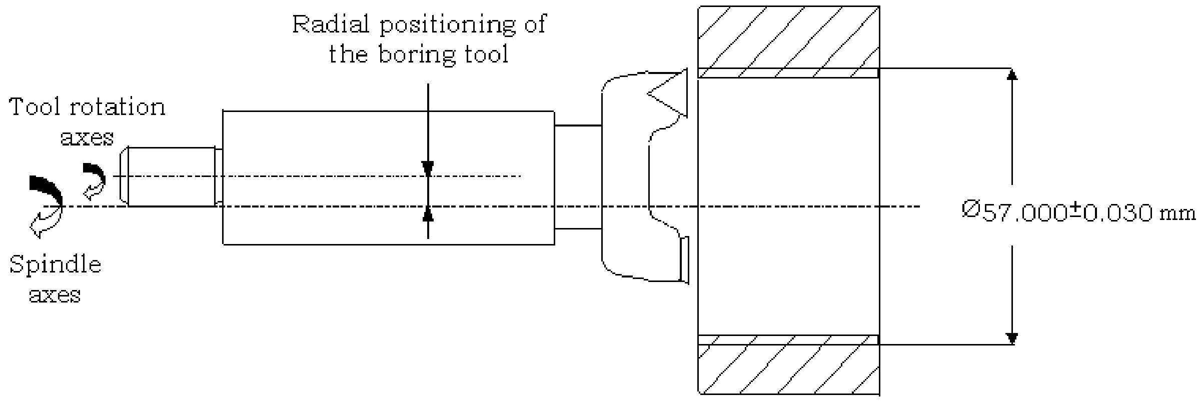

Figure 3: The hole finishing operation.

of Mean Square Error), which is obtained from an ANOVA analysis of historical process data after start-ups and which is equal to 10 microns (thus the process capability ratio, PCR, is 1). From the ANOVA analysis, it is also found that after setup or maintenance operations the process mean often exhibits a shift or offset, which is, on average, in the order of $3\sigma_\varepsilon$. In this case, parts are produced in lot of size 15.

The costs related to non-conforming items are different depending on whether the diameter obtained is below the Lower or above the Upper Specification Limit. Indeed, when the hole diameter is less than LSL, an additional machining operation can correct the defect by opportunely selecting the depth of cut. On the other hand, when the diameter obtained is greater than USL, the part has to be scrapped, since there is no possibility to recover the nonconforming workpiece. The cost of an undersized hole, c_1^c, is determined by considering the additional repairing operation while the cost of an oversized hole, c_2^c, is equal to the margin lost minus the value of the scrap. In this case, the asymmetric ratio r ($r = c_2^c/c_1^c$) is 6.5. If a quadratic cost model is assumed, by adopting the relation outlined in expressions (4), the same ratio between c_2^q and c_1^q can be obtained.

As showed in Figure 3, the controllable variable U_n is the radial position of the tool. In fact, by opportunely selecting this variable, the depth of cut can be changed, thus modifying the dimension of the diameter obtained. Furthermore, the adoption of a parametric part program can in principle allow for an automatic adjustment procedure: once a diameter is measured, the value of the controllable variable can be determined and transmitted to the control unit of the machining center that will process the next part accordingly. In a real-life application of an adjustment procedure, the resolution of the machine in setting the tool position should be considered in order to derive the approximation of the adjustment size. In this case a precision in the order of microns determines that we round the adjustment to zero decimal places.

Assuming the asymmetric constant cost function model, the expected value of the cost reported in (18) can be rewritten as a function of r, thus a scaled form of the expected costs at each step of the adjustment procedure is obtained as:

$$\frac{E(C_n^c)}{c_1^c} = \Phi\left(\frac{LSL-\mu_n}{\sigma_n}\right) + r\left[1-\Phi\left(\frac{USL-\mu_n}{\sigma_n}\right)\right] \tag{26}$$

Therefore, the performance comparisons among the different control rules will be evaluated using as performance index the Scaled Average Integrated Expected Cost (SAIEC), defined as:

$$SAIEC^c = \frac{1}{N}\sum_{n=2}^{N}\frac{E(C_n^c)}{c_1^c} \tag{27}$$

where the index in the summation starts from 2, since the quality characteristic of the first part machined does not depend on the adjustment procedure. To define the Biased adjustment rule, the steady-state target T^c and the biased coefficients b_n need to be computed and rounded to the closest integer. Using equation (20), the steady-state target results $T^c = -3$ micron. Therefore, according to (21), the biased coefficients are given by:

$$b_n = \begin{cases} 3 & n=1 \\ 0 & n=2,...,15 \end{cases} \tag{28}$$

Figure 4 reports the plots of the expected value of the quality characteristic obtained with both the Biased and Grubbs' procedures. In particular, the piecewise behavior of the biased mean converging to the target value is due to the approximation (rounding) adopted to consider the precision of the machine in setting the tool position. In fact, changing the precision of the approximation to the second decimal place, the mean at each step of the Biased procedure is represented by the dotted line in Figure 4. As it can be observed, the adoption of the Biased procedure induces a convergence of the mean to the steady-state target value T^c from the side of lower nonconforming costs.

The savings in cost obtained by the Biased rule are shown in Figure 5, where the percentage difference in $SAIEC^c$ determined by the Biased and Grubbs' procedures is reported as a function of the items processed (computed from data in Table 1).

A further comparison between the Biased and different EWMA control rules, characterized by values of the parameter λ ranging from 0.2 to 0.8 (Box and Luceño, 1995), has been carried out. Since the performance of an EWMA controller depends on the initial offset d, a constant $A = (d - T^\bullet)/\sigma_\varepsilon$, i.e., the difference between d and the target value in standard deviation units is assumed equal to 3 according to the practical case we have discussed.

The Scaled Average Integrated Expected Costs $SAIEC^c$ obtained with the Biased procedure and the EWMA controllers are reported in Table 1 and plotted in Figure 6. As it can be observed, the Biased procedures has the smallest expected cost compared to all the EWMA controllers and the advantage reduces as λ increases, So a value $\lambda = 0.8$ was used in the next comparison. It should

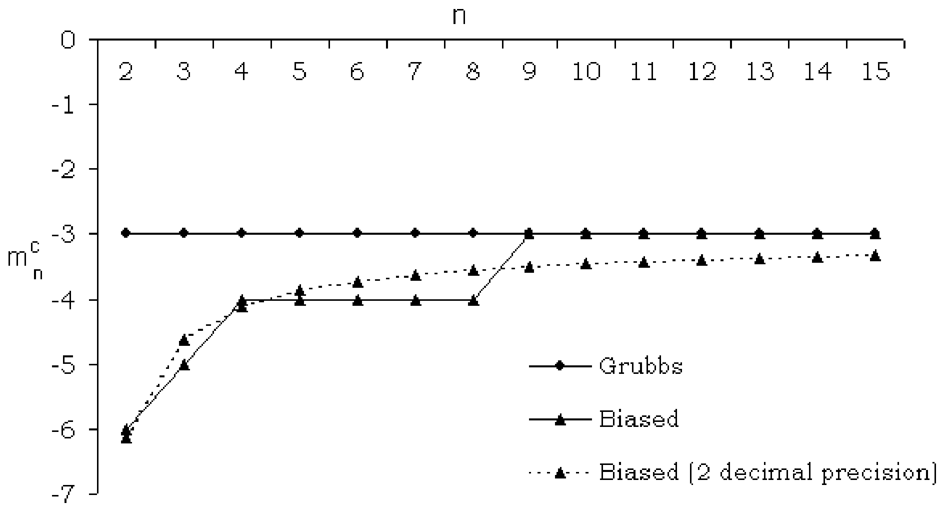

Figure 4: Trajectory of the optimal mean of the quality characteristic (m_n^c) using Grubbs' and the Biased procedures (considering 0 and 2 decimal places) under the constant cost model ($r = 6.5$ and $N = 15$).

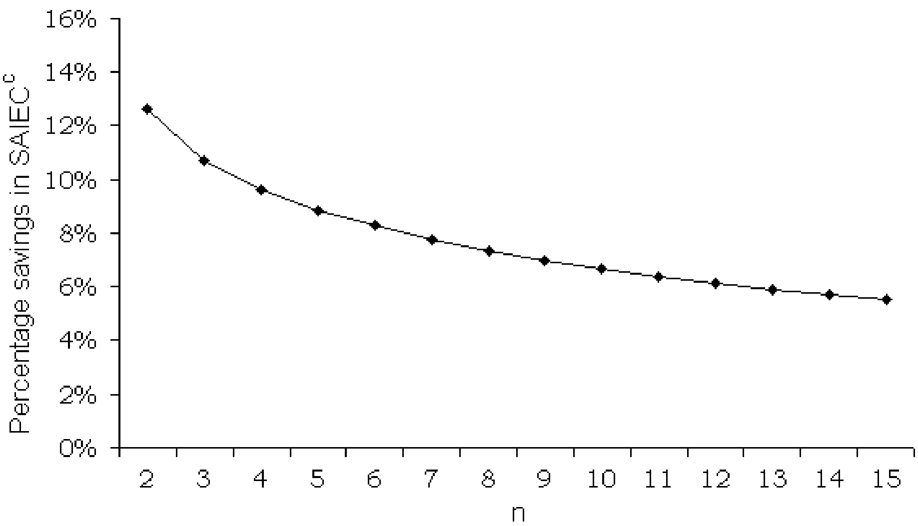

Figure 5: The percentage savings in $SAIEC^c$ ($\frac{SAIEC_G^c - SAIEC_B^c}{SAIEC_G^c} \times 100$) obtained by using the Biased procedure compared to Grubbs' rule under the constant cost function model ($r = 6.5$ and $N = 15$).

n	EWMA0.2	EWMA0.4	EWMA0.6	EWMA0.8	Grubbs	Biased
2	1.227	0.532	0.234	0.119	0.092	0.080
3	0.898	0.341	0.149	0.088	0.064	0.057
4	0.689	0.245	0.112	0.076	0.051	0.046
5	0.553	0.192	0.092	0.070	0.043	0.039
6	0.461	0.158	0.080	0.067	0.037	0.034
7	0.393	0.135	0.072	0.065	0.033	0.031
8	0.342	0.118	0.066	0.063	0.030	0.028
9	0.303	0.106	0.062	0.062	0.028	0.026
10	0.271	0.096	0.058	0.061	0.026	0.024
11	0.246	0.089	0.055	0.060	0.024	0.023
12	0.225	0.082	0.053	0.059	0.023	0.022
13	0.207	0.077	0.051	0.059	0.022	0.021
14	0.193	0.072	0.050	0.058	0.021	0.020
15	0.180	0.068	0.048	0.058	0.020	0.019

Table 1: The $SAIEC^c$ adopting different control rules ($r = 6.5$, $N = 15$ and $A = 3$).

be pointed out that much smaller values of λ are recommended in the literature (Box and Luceño, 1995), but for these values of λ the EWMA performs relatively worse.

The cost comparison between the EWMA controller with $\lambda = 0.8$ and the Biased controller is given in Figure 7, where the percentage saving in $SAIEC^c$ induced by the Biased procedure over the EWMA is plotted. It is interesting to find that the advantage induced by the Biased procedure is even higher as the number of parts produced increases. The reason for this behavior lies on the long-term performance of the EWMA control rule. In fact, as n tends to infinity, the mean of the Y_n regulated by the EWMA controller approaches zero, but the variance approaches to the value $2\sigma_\varepsilon^2/(2-\lambda)$, which is greater than σ_ε^2. This inflation in variance has been discussed by Box and Luceño (1997) and del Castillo (2001).

For the quadratic cost function model, an analogous comparison was performed. In this case, the expected cost reported in equation (22) can be rewritten in scaled form by manipulating the expression as follows:

$$E(C_n^q) = c_1^q \sigma_n^2 \left\{ \frac{c_2^q}{c_1^q} \left(\frac{\mu_n^2}{\sigma_n^2} + 1 \right) + \left(\frac{c_2^q}{c_1^q} - 1 \right) \left[\frac{\mu_n}{\sigma_n} \phi\left(\frac{\mu_n}{\sigma_n} \right) - \left(\frac{\mu_n^2}{\sigma_n^2} + 1 \right) \Phi\left(-\frac{\mu_n}{\sigma_n} \right) \right] \right\} .$$

Considering that the variance at each step of the adjustment procedure (12) is proportional to the variance of the error σ_ε^2, the expected cost at the n^{th} step of the procedure is given by:

$$\frac{E(C_n^q)}{c_1^q} = s_n \sigma_\varepsilon^2 \left\{ r(\delta_n^2 + 1) + (r-1) \left[\delta_n \phi(\delta_n) - (\delta_n^2+1)\Phi(-\delta_n) \right] \right\} ,$$

where $s_n = \left[1 + \sum_{i=1}^{n-1} k_i^2 \prod_{j=i+1}^{n-1} (1-k_j)^2 \right]$ represents the ratio between σ_n^2 and σ_ε^2 in equation (12), r denotes the ratio between c_2^q and c_1^q and δ_n the ratio between μ_n and σ_n.

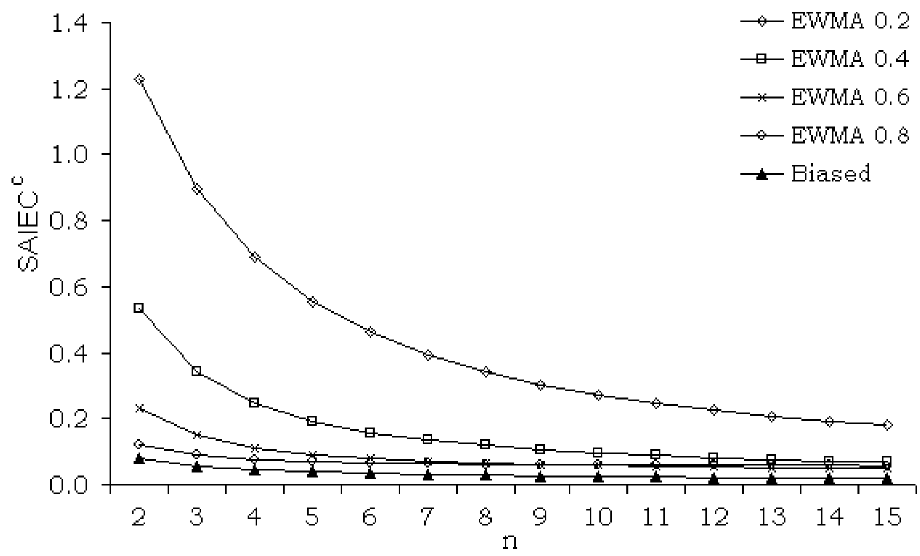

Figure 6: Comparison of $SAIEC^c$'s determined by the EWMA controllers (with different values of λ) and the Biased procedure under the constant cost function model ($r = 6.5$, $N = 15$ and $A = 3$).

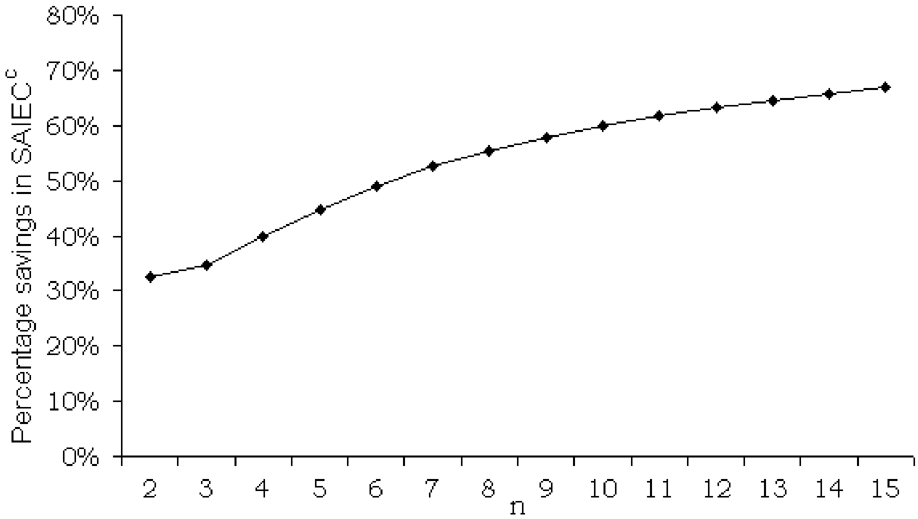

Figure 7: The percentage savings in $SAIEC^c$ ($\frac{SAIEC^c_{EWMA0.8} - SAIEC^c_B}{SAIEC^c_{EWMA0.8}} \times 100$) obtained by using the Biased procedure compared to the EWMA rule with $\lambda = 0.8$ under the constant cost function model ($r = 6.5$, $N = 15$ and $A = 3$).

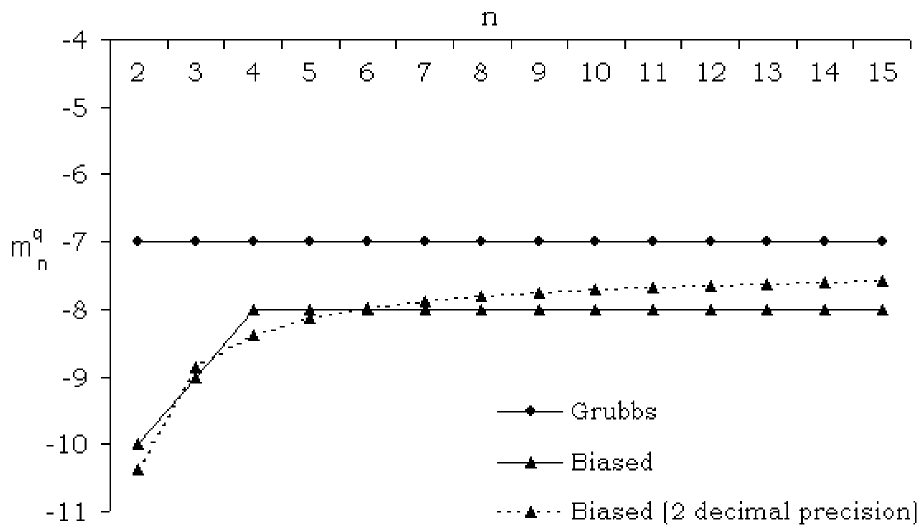

Figure 8: Trajectory of the optimal mean of the quality characteristic (m_n^q) using Grubbs' and the Biased procedures (considering 0 and 2 decimal places) under the quadratic cost model ($r = 6.5$ and $N = 15$).

As in the constant cost function case, the performance index considered is related to the Scaled Average Integrated Expected Cost defined as:

$$SAIEC^q = \frac{1}{N}\sum_{n=2}^{N}\frac{E(C_n^q)}{c_1^q} \tag{29}$$

Figure 8 reports the plot of the mean of the quality characteristic obtained with Grubbs' rule (in which the mean is constant and equal to the steady-state target value), and the Biased rule (in which the mean is set to m_n^q and converges to the target value). Similarly as in the constant cost model case, the mean induced by the Biased procedure is computed by considering the assumption on the control variable resolution (in Figure 8 the theoretical behavior of one-sided convergence of m_n^q is reported with a dotted line, which was obtained by rounding m_n^q to the second decimal place). The values of the biased coefficients b_n are also shown in Table 2. As it can be observed, when the precision of the machine is considered, the sequence $\{b_n\}$ adopted is basically the same as obtained with the asymmetric constant cost model (28), but the computation of b_n in the constant cost model is much easier because of the closed form expressions. Therefore, this numerical result permits to outline an approximated way to compute the b_n that does not require the numerical solution of equation (23).

Data on the $SAIEC^q/\sigma_\varepsilon^2$ obtained with the Grubbs' rule, the Biased rule and the EWMA controller are reported in Table 3. The percentage in savings from adopting the Biased procedure instead of Grubbs' rule are reported in Figure 9. Figures 10 and 11 report respectively the $SAIEC^q/\sigma_\varepsilon^2$ obtained with the Biased procedure and the EWMA controllers and the detail on the

n	b_n	b_n (2 decimal precision)
1	3	3.06
2	0	0.26
3	0	0.11
4	0	0.06
5	0	0.04
6	0	0.03
7	0	0.02
8	0	0.01
9	0	0.01
10	0	0.01
11	0	0.01
12	0	0.01
13	0	0.01
14	0	0.00
15	0	0.00

Table 2: The bias coefficients b_n computed under the quadratic cost function rounding to the nearest integer or considering the second decimal place ($r = 6.5$ and $N = 15$).

n	EWMA0.2	EWMA0.4	EWMA0.6	EWMA0.8	Grubbs	Biased
2	25.461	14.947	8.674	5.625	4.818	4.539
3	20.721	11.046	6.545	4.851	4.158	3.970
4	17.300	8.761	5.560	4.544	3.798	3.657
5	14.829	7.452	5.028	4.391	3.566	3.451
6	13.035	6.579	4.685	4.298	3.402	3.305
7	11.635	5.974	4.457	4.237	3.280	3.195
8	10.505	5.543	4.294	4.193	3.184	3.109
9	9.614	5.219	4.172	4.160	3.107	3.039
10	8.889	4.958	4.077	4.135	3.043	2.982
11	8.287	4.749	4.001	4.114	2.990	2.933
12	7.794	4.579	3.939	4.098	2.944	2.892
13	7.370	4.436	3.887	4.084	2.904	2.856
14	7.011	4.316	3.843	4.072	2.869	2.824
15	6.696	4.213	3.805	4.062	2.838	2.796

Table 3: The $SAIEC^q/\sigma_\varepsilon^2$ adopting different control rules ($r = 6.5$, $N = 15$ and $A = 3$)

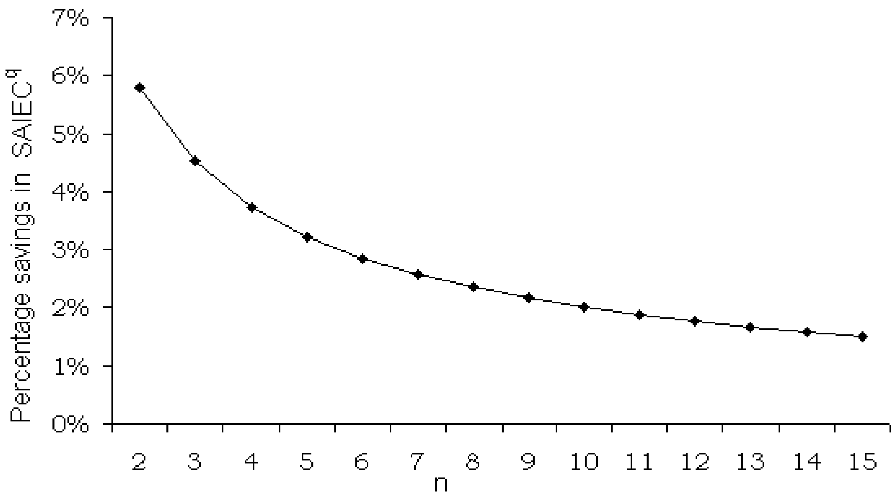

Figure 9: The percentage savings in $SAIEC^q$ ($\frac{SAIEC_G^q - SAIEC_B^q}{SAIEC_G^q} \times 100$) obtained by using the Biased procedure compared to Grubbs' rule under the quadratic cost function model ($r = 6.5$ and $N = 15$).

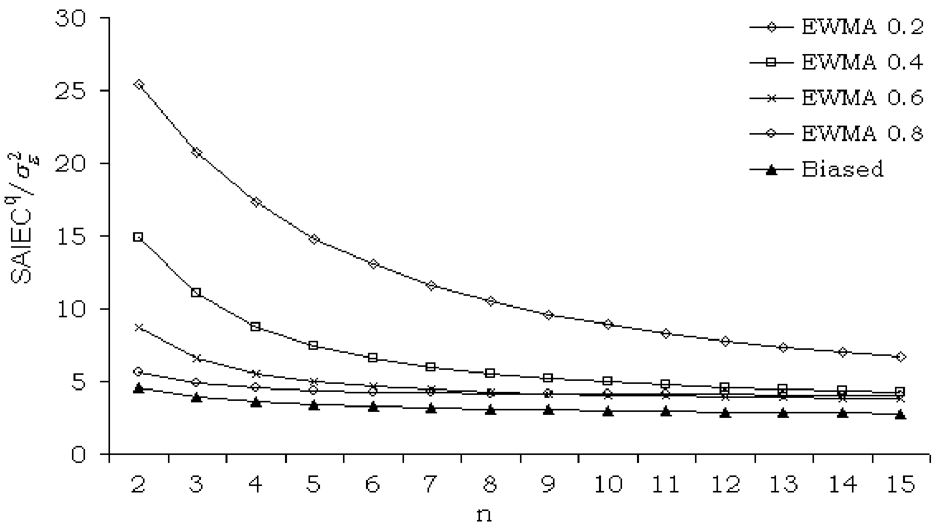

Figure 10: Comparison of $SAIEC^q/\sigma_\varepsilon^2$'s determined by the EWMA controllers (with different values of λ) and the Biased procedure under the quadratic cost function model ($r = 6.5$, $N = 15$ and $A = 3$).

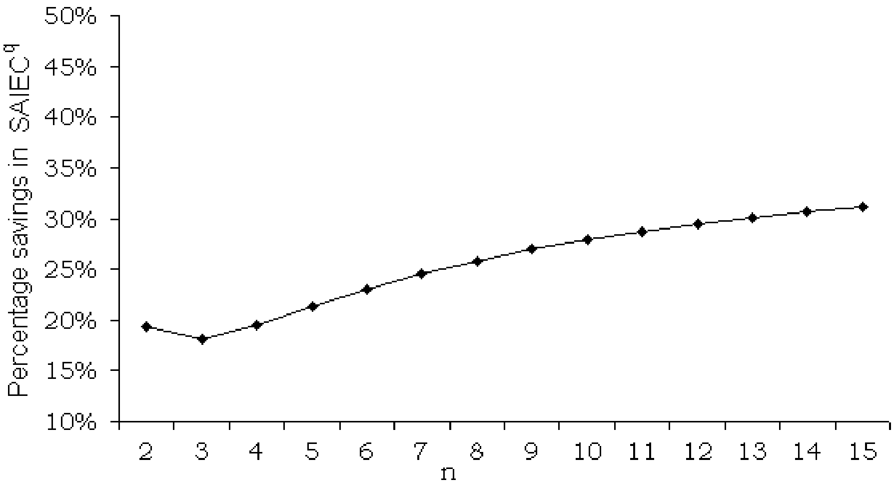

Figure 11: The percentage savings in $SAIEC^q$ ($\frac{SAIEC^q_{EWMA0.8}-SAIEC^q_B}{SAIEC^q_{EWMA0.8}} \times 100$) obtained by using the Biased procedure compared to the EWMA rule with $\lambda = 0.8$ under the quadratic cost function model ($r = 6.5$, $N = 15$ and $A = 3$).

percentage savings obtained over the EWMA controller with $\lambda = 0.8$.

The comparisons between the Biased control rule, Grubbs' procedure and the EWMA indicate the same conclusions as for the constant asymmetric cost model, but quantitatively, the magnitude of the percentage advantage obtained with the Biased rule is greater when adopting the constant cost model.

6 Sensitivity Analysis

A numerical comparison of the performance obtained with the Biased procedure, Grubbs' rule and the EWMA controllers was conducted to characterize situations in which the adoption of the feedback adjustment could be more profitable. The comparison has been carried out first for the Biased procedure versus Grubbs' rule, since the performance in this case does not depend on the initial offset. The variables affecting the results in this case are the coefficient r, representing the asymmetry of the cost function, and N, the number of parts processed in each lot. The value of r was varied from 1 to 11 as in Ladany (1995). We point out that two real cases of asymmetric cost functions considered in Wu and Tang (1998) and Moorhead and Wu (1998) have r to be 4 and 6, respectively, and they are inside the range examined. The number of parts in the lot, N, was varied from 1 to 40.

Figures 12 and 13 present the savings in cost obtained with the Biased procedure over Grubbs' rule for the constant and quadratic cost models, respectively. As it can be observed, the Biased procedure has an advantage especially on the first parts produced (this suggests the adoption of the

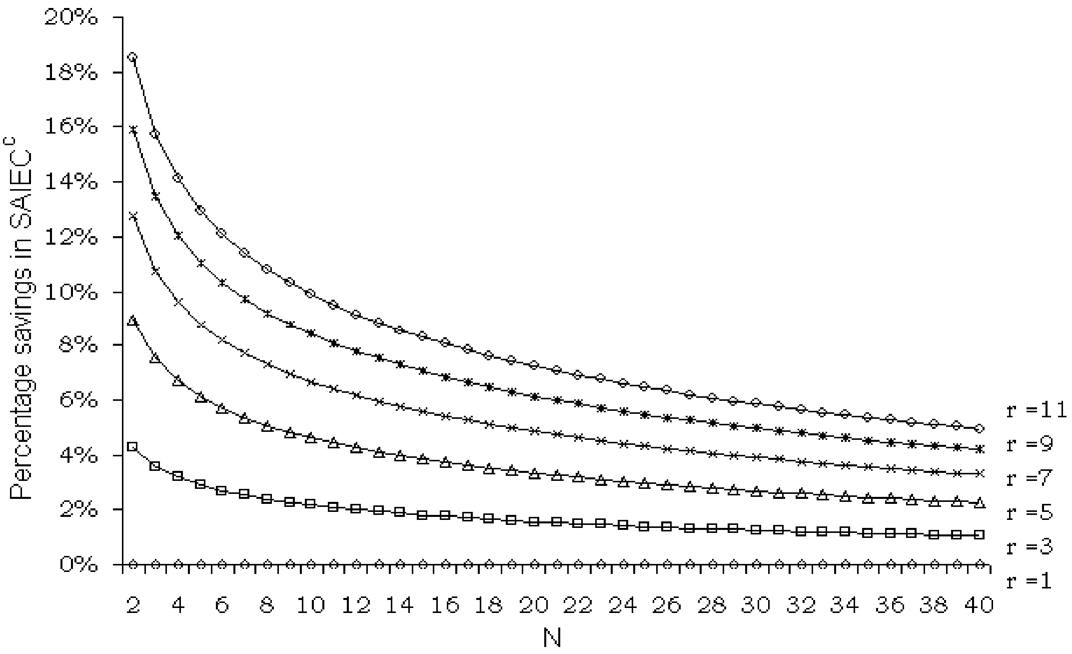

Figure 12: Sensitivity analysis: the percentage saving in $SAIEC^c$ ($\frac{SAIEC_G^c - SAIEC_B^c}{SAIEC_G^c} \times 100$) obtained by using the Biased procedure compared to Grubbs' procedure under the constant cost function model, when the asymmetry ratio is varied.

Biased rule when parts are produced in small lots) and this advantage increases as the asymmetry in the function becomes more evident (i.e., as r increases).

Since the performance of the EWMA controllers depend on the initial offset d, standardized by the constant $A = (d - T^\bullet)/\sigma_\varepsilon$, the comparison between the Biased rule and the EWMA controller has been performed by considering A ranging from -4 to 4. Figures 14 and 15 report the difference in the Scaled Average Integrated Expected costs obtained with the EWMA and the Biased controller, under the constant and the quadratic cost models, respectively. In particular, the difference is reported for the two extreme values of λ (i.e., $\lambda = 0.2$ and $\lambda = 0.8$) and the lot size (i.e., $N = 5$ and $N = 40$).

Depending on the initial offset, the advantage of using the Biased procedure varies dramatically. Considering the case in which $\lambda = 0.2$, when A is greater than 1 the performance of the Biased procedure dominates that of the EWMA controller, but the difference between the two procedures is almost negligible as A is close to zero. Furthermore, the advantage is asymmetric too. In particular, if A is positive, i.e. the offset d arises from the side in which non-conforming items are more expensive, the advantage of adopting the Biased procedure is significantly greater, compared with the case in which the initial shift has the same magnitude but different sign. As the number of parts processed in the lot increases, the difference between the two procedures maintains the same

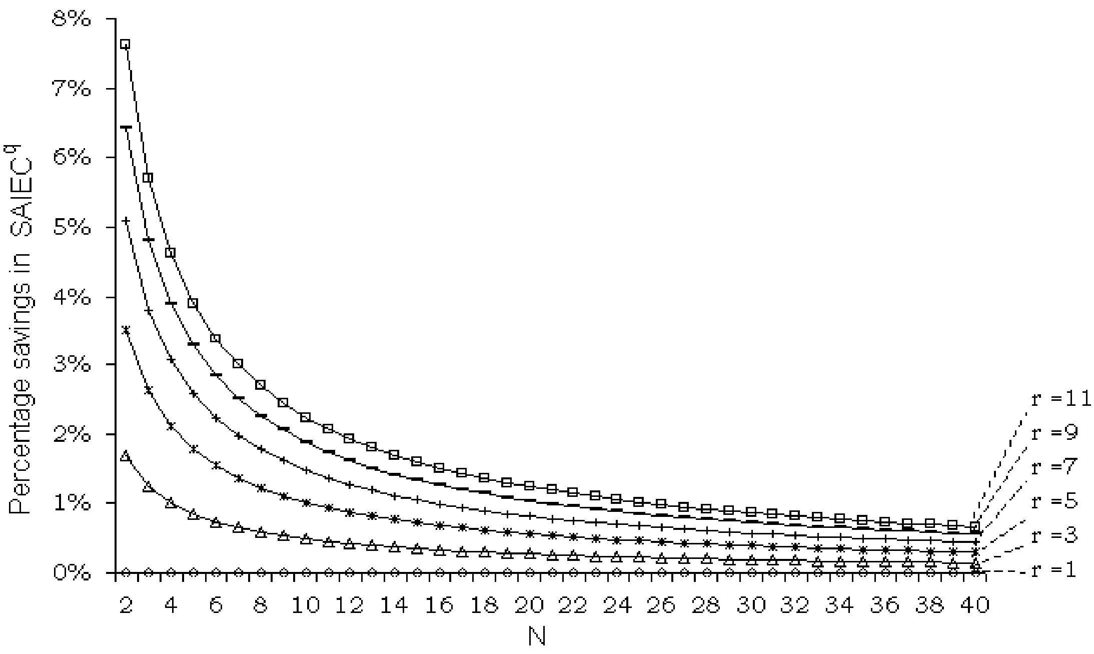

Figure 13: Sensitivity analysis: the percentage saving in $SAIEC^q$ ($\frac{SAIEC_G^q - SAIEC_B^q}{SAIEC_G^q} \times 100$) obtained by using the Biased procedure compared to Grubbs' rule under the quadratic cost function model, when the asymmetry ratio is varied.

behavior while reducing in magnitude (both approaches tend to reach their asymptotic performance, which are different only with respect to the variance σ_n^2 of the quality characteristic).

In the case when $\lambda = 0.8$ is used in the EWMA control rule, the advantage determined by the Biased approach is reduced but is always greater than zero, regardless of the direction of the initial offset. Also in this case, the effect of A becomes even less significant when N increases. The problem with the EWMA controller, of course, is that it is not clear how to choose λ.

7 Conclusions

The problem of designing an adjustment rule to correct a process start-up error has recently received a renewed attention in the literature. This attention is related to two tendencies that are shown in modern manufacturing. The first one is the adoption of small lot sizes, which leads to an increase in the number of setups required on the machine. The second one is the growing frequency in changing product specifications, which increases the chance of systematic errors at the start-up of a manufacturing process. Therefore, applying feedback adjustments for process start-up errors becomes an effective way to reduce the number of non-conforming parts. Up to now, previous approaches to setup adjustment problems have only considered symmetric cost functions. This paper presented a feedback adjustment rule that can be adopted when an asymmetric cost model

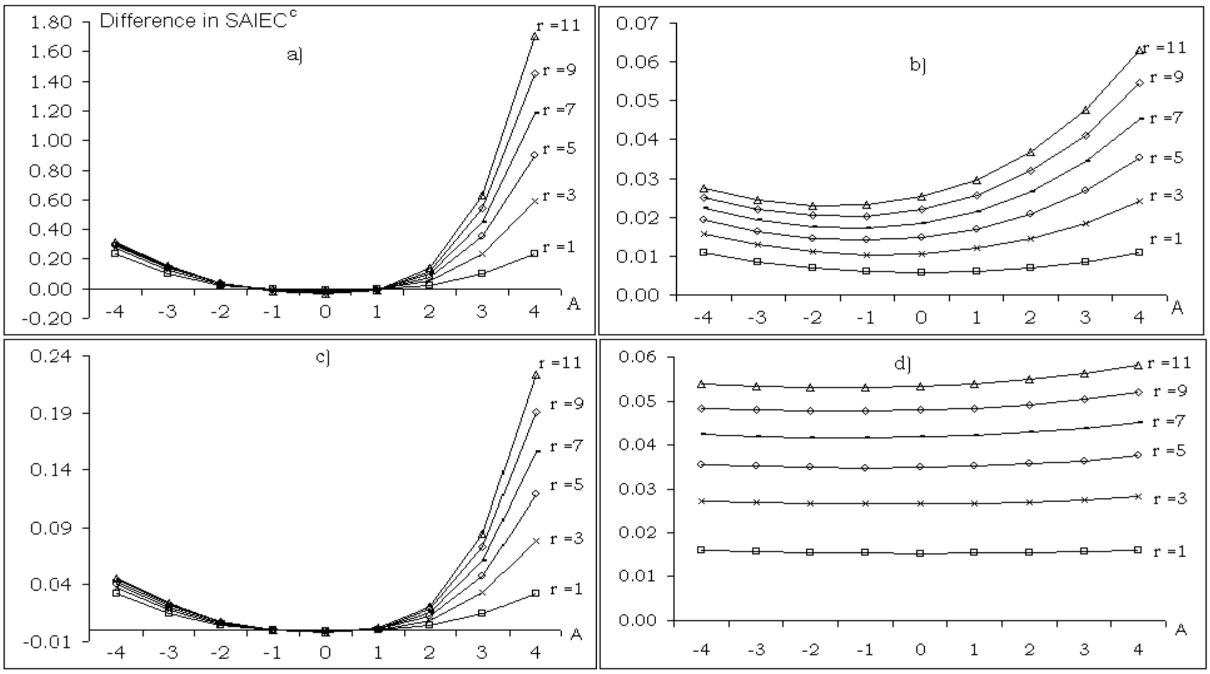

Figure 14: Sensitivity analysis: the difference in $SAIEC^c$ ($SAIEC^c_{EWMA} - SAIEC^c_B$) obtained by using the EWMA controller and the Biased rule under the constant cost model, when r, A and N are varied. a)$\lambda = 0.2$ and $N = 5$; b) $\lambda = 0.8$ and $N = 5$; c) $\lambda = 0.2$ and $N = 40$; d) $\lambda = 0.8$ and $N = 40$.

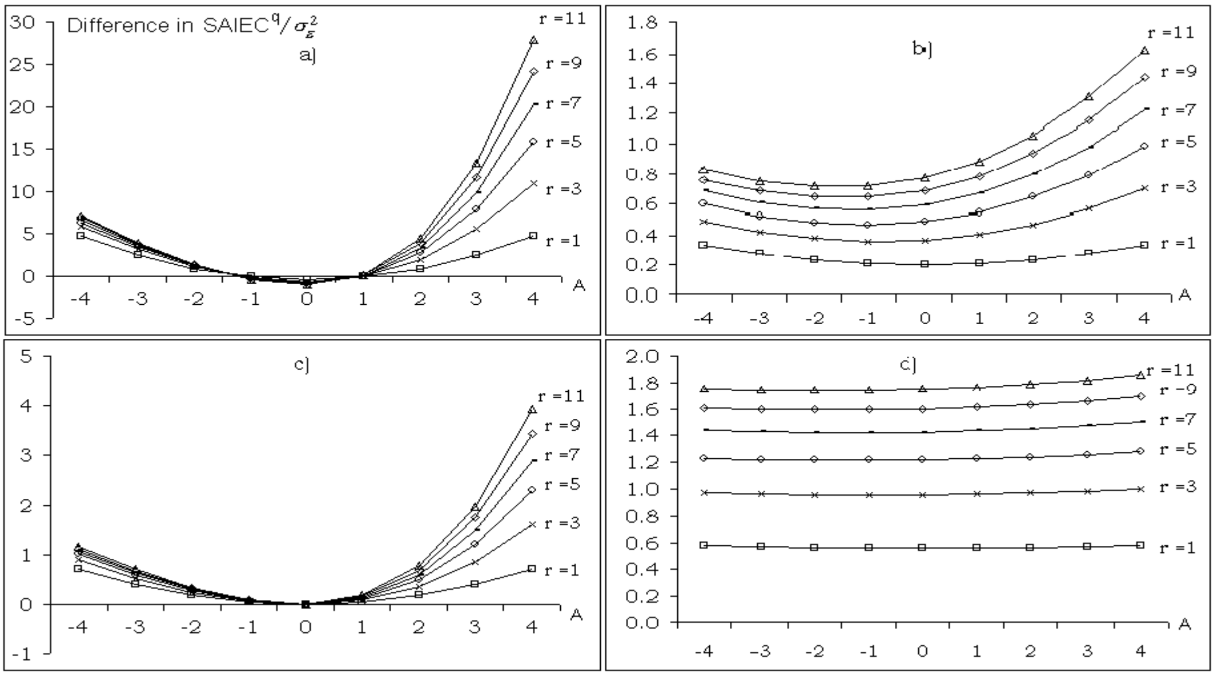

Figure 15: Sensitivity analysis: the difference in $SAIEC^q/\sigma_\varepsilon^2$ ($\frac{SAIEC^q_{EWMA}-SAIEC^q_B}{\sigma_\varepsilon^2}$) obtained by using the EWMA controller and the Biased under the quadratic cost model, when r, A and N are varied. a)$\lambda = 0.2$ and $N = 5$; b) $\lambda = 0.8$ and $N = 5$; c) $\lambda = 0.2$ and $N = 40$; d) $\lambda = 0.8$ and $N = 40$.

can better represent the process quality losses entailed. In particular, two asymmetric cost functions that are often encountered in manufacturing have been considered. In the first case, the cost of a non-conforming item is assumed constant but changes depending whether the quality characteristic is below the Lower or above the Upper Specification Limit. In the second case, costs are supposed to be proportional to the square of the distance of the quality characteristic from the nominal value, but the proportional constant is allowed to change with the sign of this difference.

Starting from the general form of a linear controller, the biased feedback adjustment rule has been derived by minimizing all the costs incurred during the transient phase in which the quality characteristic converges to its steady-state target. A numerical comparison of the cost incurred by the adjustment rule proposed and other rules assessed in the literature showed that the proposed procedure is effective, especially when the asymmetry in the cost function or the initial process offset are significant. Compared to Grubbs' rule, the proposed Biased adjustment rule is recommended especially for manufacturing expensive parts which usually are produced in small lots (e.g., in the aerospace industry).

Besides the two specific cost functions studied herein, the proposed approach can be easily extended to deal with different production situations in which the cost function is asymmetric, such as a piece-wise linear function used in filling processes (Misiorek and Barnett, 2000).

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Appendix A.1: Equivalence between the minimization of $AIEC^\bullet$ and the minimization of the expected cost $E(C_n^\bullet)$

Consider the minimization problem

$$min_{\boldsymbol{\mu}}\ AIEC^\bullet$$

where $\boldsymbol{\mu} = \{\mu_n,\ n = 1, ..., N\}$ is the $N \times 1$ vector of the means of the quality characteristic and $AIEC^\bullet$ is given by (5). The assumption of linear feedback adjustment rule induces an affine relation among the means of the response variable, which can be generally expressed as $\boldsymbol{\mu} = \boldsymbol{R}\boldsymbol{\mu} + \boldsymbol{s}$, where

$\boldsymbol{R}$ is a $N \times N$ matrix and $\boldsymbol{s}$ is a $N \times 1$ vector. In particular the mean at the n^{th} step μ_n can be derived as:

$$\mu_n = \sum_{i=1}^{N} r_{ni}\mu_i + s_n \tag{30}$$

where r_{ni} is the entry in row n and column i of the matrix $\boldsymbol{R}$ and s_n is the n^{th} component of $\boldsymbol{s}$. In particular, the mean at each step is a function only of the previous ones, therefore $r_{ni} = 0$ for $i \geq n$.

To minimize $AIEC^{\bullet}$ the first order condition consists in equating to zero all the components of the gradient vector, i.e.:

$$\boldsymbol{\nabla} AIEC^{\bullet} = \mathbf{0} \ ,$$

where $\mathbf{0}$ is a $N \times 1$ vector of zeros. Considering the expression of $AIEC^{\bullet}$ given by (5), the i^{th} component of the gradient can be rewritten as:

$$\frac{\partial AIEC^{\bullet}}{\partial \mu_i} = \frac{1}{N}\sum_{n=1}^{N}\frac{\partial E(C_n^{\bullet})}{\partial \mu_i} = \frac{1}{N}\sum_{n=1}^{N}\frac{\partial E(C_n^{\bullet})}{\partial \mu_n}\frac{\partial \mu_n}{\partial \mu_i} = \frac{1}{N}\sum_{n=1}^{N}\frac{\partial E(C_n^{\bullet})}{\partial \mu_n} r_{ni} \ .$$

Therefore, the first order condition is satisfied when

$$\frac{\partial E(C_n^{\bullet})}{\partial \mu_n} = 0 \ , \ n = 1, 2, ..., N \ . \tag{31}$$

The second order condition can be determined by considering two theorems, derived by extending to strictly convex functions results reported in (Bazaraa *et al.*, 1993) for convex functions.

Theorem 1

Let $f_1, f_2, ..., f_k : E_n \to E_1$ be strictly convex functions. Then, the function f defined as $f(\mathbf{x}) = \sum_{j=1}^{k} \alpha_j f_j(\mathbf{x})$, where $\alpha_j > 0$ for $j = 1, ..., k$ is strictly convex.

Theorem 2

Let $g : E_m \to E_1$ be a strictly convex function and let $\mathbf{h}$: $E_n \to E_m$ be an affine function of the form $\mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is an $m \times 1$ vector. Then, the composite function $f : E_n \to E_1$, defined as $f(\mathbf{x}) = g\left[\mathbf{h}(\mathbf{x})\right]$, is strictly convex.

Considering expression (5), $AIEC^{\bullet}$ can be seen as a linear combination of $E(C_n^{\bullet})$ with weights $1/N$. Therefore the first theorem allows to assert that $AIEC^{\bullet}$ is strictly convex when each component $E(C_n^{\bullet})$ is also a strictly convex function of the vector of means $\boldsymbol{\mu}$. On the other hand, the expected cost at time n, $E(C_n^{\bullet})$, is a composite function, since it is directly related only to one component of the vector, namely μ_n, which in turn depends on the whole vector $\boldsymbol{\mu}$ through an affine relation (given by equation 30). Therefore, considering the second theorem, the strictly convexity of $E(C_n^{\bullet})$ as a function of the whole vector $\boldsymbol{\mu}$, is proved once it is showed that $E(C_n^{\bullet})$ is a strictly convex function of the scalar μ_n. Merging the results from the first and second theorems, the second order condition can be stated as:

$$\frac{\partial^2 E(C_n^{\bullet})}{\partial^2 \mu_n} > 0 \ . \tag{32}$$

This condition implies that $AIEC^{\bullet}$ is a strictly convex function, thus characterized by a unique and global minimum. Considering both the first and the second order conditions, given respectively by (31) and (32), the minimization of $AIEC^{\bullet}$ considered can be replaced by the following set of minimization problems:

$$min_{\mu_n}\ E(C_n^{\bullet})\ ,\ n = 1, 2, ..., N\ .$$

Appendix A.2: Minimization of the expected costs $E(C_n^c)$ for the asymmetric constant cost function

In the case of the constant cost function

$$E(C_n^c) = c_1^c \Phi\left(\frac{LSL - \mu_n}{\sigma_n}\right) + c_2^c\left[1 - \Phi\left(\frac{USL - \mu_n}{\sigma_n}\right)\right]\ ,$$

taking the first derivative with respect to μ_n and equating it to zero, we get

$$\frac{\partial}{\partial \mu_n}E(C_n^c) = -\frac{c_1^c}{\sigma_n}\phi\left(\frac{LSL - \mu_n}{\sigma_n}\right) + \frac{c_2^c}{\sigma_n}\phi\left(\frac{USL - \mu_n}{\sigma_n}\right) = 0\ .$$

Therefore, the condition for the optimal target at time n, m_n^c, is given by

$$\frac{\phi\left(\frac{USL - m_n^c}{\sigma_n}\right)}{\phi\left(\frac{LSL - m_n^c}{\sigma_n}\right)} = \frac{c_1^c}{c_2^c}\ . \tag{33}$$

Considering the analytical expression of the normal density $\phi(\cdot)$, equation (33) can be rewritten as:

$$\frac{\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{USL - m_n^c}{\sigma_n}\right)^2\right]}{\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{LSL - m_n^c}{\sigma_n}\right)^2\right]} = \exp\left\{-\frac{1}{2}\left[\left(\frac{USL - m_n^c}{\sigma_n}\right)^2 - \left(\frac{LSL - m_n^c}{\sigma_n}\right)^2\right]\right\} = \frac{c_1^c}{c_2^c} \tag{34}$$

By taking the logarithm on both sides of equation (34), the closed form expression of the optimal target m_n^c can be obtained as follows:

$$\frac{USL^2 + m_n^{c\ 2} - 2\ USL\ m_n^c - LSL^2 - m_n^{c\ 2} + 2\ LSL\ m_n^c}{\sigma_n^2} = -2\ln\left(\frac{c_1^c}{c_2^c}\right)$$

$$USL^2 - LSL^2 - 2(USL - LSL)m_n^c = -2\sigma_n^2\ln\left(\frac{c_1^c}{c_2^c}\right)$$

$$m_n^c = \frac{2\sigma_n^2\ln(\frac{c_1^c}{c_2^c}) + USL^2 - LSL^2}{2(USL - LSL)} = \frac{\sigma_n^2\ln(\frac{c_1^c}{c_2^c})}{(USL - LSL)} + \frac{1}{2}(USL + LSL)\ . \tag{35}$$

In order to evaluate if the optimal mean m_n^c obtained determines a minimum of the cost function, the second order derivative has to be considered:

$$\begin{aligned}
\frac{\partial^2}{\partial \mu_n^2} E(C_n^c) &= \frac{\partial}{\partial \mu}\left[-\frac{c_1^c}{\sigma_n}\phi\left(\frac{LSL-\mu_n}{\sigma_n}\right)+\frac{c_2^c}{\sigma_n}\phi\left(\frac{USL-\mu_n}{\sigma_n}\right)\right] \\
&= \frac{c_1^c}{\sigma_n^3}(\mu_n - LSL)\phi\left(\frac{LSL-\mu_n}{\sigma_n}\right)+\frac{c_2^c}{\sigma_n^3}(USL-\mu_n)\phi\left(\frac{USL-\mu_n}{\sigma_n}\right) \ . \qquad (36)
\end{aligned}$$

As it can be observed, this is always greater than zero as long as the condition $LSL < \mu_n < USL$ is satisfied.

Appendix A.3: The Expected Costs $E(C_n^q)$ for the asymmetric quadratic cost function

Consider the expected cost at time n given by:

$$E(C_n^q) = c_1^q \int_{-\infty}^{0} y_n^2 f_N(y_n;\mu_n,\sigma_n^2)dy_n + c_2^q \int_{0}^{\infty} y_n^2 f_N(y_n;\mu_n,\sigma_n^2)dy_n \ . \qquad (37)$$

Since c_1^q and c_2^q are constants, the expression of the expected value of cost at time n is completely defined by solving the generic integral:

$$\int_a^b y^2 f_N(y;\mu,\sigma^2)dy = \frac{1}{\sqrt{2\pi}\sigma}\int_a^b y^2 \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]dy \ . \qquad (38)$$

Let $z = \frac{y-\mu}{\sigma}$, thus $y = \mu + \sigma z \rightarrow dy = \sigma dz$, $y = a \rightarrow z = \frac{a-\mu}{\sigma} = c$ and $y = b \rightarrow z = \frac{b-\mu}{\sigma} = d$. Hence, the integral in (38) can be rewritten as:

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}\sigma}\int_c^d (\mu+\sigma z)^2 \exp\left(-\frac{z^2}{2}\right)\sigma dz = \\
&\frac{1}{\sqrt{2\pi}}\left\{\mu^2\int_c^d \exp\left(-\frac{z^2}{2}\right)dz + \sigma^2\int_c^d z^2\exp\left(-\frac{z^2}{2}\right)dz + 2\mu\sigma\int_c^d z\exp\left(-\frac{z^2}{2}\right)dz\right\} \ . \qquad (39)
\end{aligned}$$

The first term on the right hand side of (39) can be simply calculated as:

$$\frac{\mu^2}{\sqrt{2\pi}}\int_c^d \exp\left(-\frac{z^2}{2}\right)dz = \mu^2\left[\Phi(d)-\Phi(c)\right] \ ,$$

where $\Phi(\cdot)$ represents the cumulative standard normal distribution function. The second term can be evaluated integrating by parts as follows:

$$\begin{aligned}
\frac{\sigma^2}{\sqrt{2\pi}}\int_c^d z^2 \exp\left(-\frac{z^2}{2}\right) dz &= -\frac{\sigma^2}{\sqrt{2\pi}}\int_c^d z(-z)\exp\left(-\frac{z^2}{2}\right) dz \\
&= -\frac{\sigma^2}{\sqrt{2\pi}}\int_c^d z\left[\frac{d}{dz}\exp\left(-\frac{z^2}{2}\right)\right] dz \\
&= -\frac{\sigma^2}{\sqrt{2\pi}}\left[z\exp\left(-\frac{z^2}{2}\right)\right]_c^d + \frac{\sigma^2}{\sqrt{2\pi}}\int_c^d \exp\left(-\frac{z^2}{2}\right) dz \\
&= -\frac{\sigma^2}{\sqrt{2\pi}}\left[d\exp\left(-\frac{d^2}{2}\right) - c\exp\left(-\frac{c^2}{2}\right)\right] + \sigma^2\left[\Phi(d)-\Phi(c)\right] \quad .
\end{aligned}$$

Finally, the third term in (39) can be computed as follows:

$$\begin{aligned}
\frac{2\mu\sigma}{\sqrt{2\pi}}\int_c^d z\exp\left(-\frac{z^2}{2}\right) dz &= -\frac{2\mu\sigma}{\sqrt{2\pi}}\int_c^d (-z)\exp\left(-\frac{z^2}{2}\right) dz \\
&= -\frac{2\mu\sigma}{\sqrt{2\pi}}\int_c^d \left[\frac{d}{dz}\exp\left(-\frac{z^2}{2}\right)\right] dz \\
&= -\frac{2\mu\sigma}{\sqrt{2\pi}}\left[\exp\left(-\frac{z^2}{2}\right)\right]_c^d \\
&= -\frac{2\mu\sigma}{\sqrt{2\pi}}\left[\exp\left(-\frac{d^2}{2}\right) - \exp\left(-\frac{c^2}{2}\right)\right] = -2\mu\sigma\left[\phi(d)-\phi(c)\right] \quad .
\end{aligned}$$

Therefore:

$$\begin{aligned}
\int_a^b y^2 f_N(y;\mu,\sigma^2) dy &= (\mu^2+\sigma^2)\left[\Phi(d)-\Phi(c)\right] + \\
&\quad -\frac{\sigma^2}{\sqrt{2\pi}}\left[d\exp\left(-\frac{d^2}{2}\right) - c\exp\left(-\frac{c^2}{2}\right)\right] + -2\mu\sigma\left[\phi(d)-\phi(c)\right] \ ,
\end{aligned} \tag{40}$$

where $\frac{a-\mu}{\sigma} = c$ and $\frac{b-\mu}{\sigma} = d$. With this result, the first integral in (37) can be computed by evaluating (40) when $c \to -\infty$ and $d = -\frac{\mu}{\sigma}$. We have that:

$$\begin{aligned}
\int_{-\infty}^0 y_n^2 f_N(y_n;\mu_n,\sigma_n^2) dy_n &= (\mu_n^2+\sigma_n^2)\Phi\left(-\frac{\mu_n}{\sigma_n}\right) + \\
&\quad -\frac{\sigma_n^2}{\sqrt{2\pi}}\left[-\frac{\mu_n}{\sigma_n}\exp\left(-\frac{\mu_n^2}{2\sigma_n^2}\right) - \lim_{c\to-\infty} c\exp\left(-\frac{c^2}{2}\right)\right] - 2\mu_n\sigma_n\phi\left(\frac{\mu_n}{\sigma_n}\right)
\end{aligned}$$

and, by using De L'Hospital's rule:

$$\lim_{c\to-\infty} c\exp\left(-\frac{c^2}{2}\right) = \lim_{c\to-\infty}\frac{c}{\exp\left(\frac{c^2}{2}\right)} = \lim_{c\to-\infty}\frac{1}{c\exp\left(\frac{c^2}{2}\right)} = 0 \ .$$

Hence,

$$\int_{-\infty}^{0} y_n^2 f_N(y_n;\mu_n,\sigma_n^2)dy_n = (\mu_n^2+\sigma_n^2)\Phi\left(-\frac{\mu_n}{\sigma_n}\right) - \sigma_n\mu_n\phi\left(\frac{\mu_n}{\sigma_n}\right) \ .$$

The second integral in (37) can be analogously computed, considering that in this case $c = -\frac{\mu}{\sigma}$ and $d \to \infty$. This is given by:

$$\int_{0}^{\infty} y_n^2 f_N(y_n;\mu_n,\sigma_n^2)dy_n = (\mu_n^2+\sigma_n^2)\left[1-\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] + \sigma_n\mu_n\phi\left(\frac{\mu_n}{\sigma_n}\right) \ .$$

Therefore, the expected costs in equation (37) can be rewritten as:

$$E(C_n^q) = c_2^q(\mu_n^2+\sigma_n^2) + (c_2^q - c_1^q)\left[\sigma_n\mu_n\phi\left(\frac{\mu_n}{\sigma_n}\right) - (\mu_n^2+\sigma_n^2)\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] \ . \tag{41}$$

Appendix A.4: The minimization of the $E(C_n^q)$ for the asymmetric quadratic cost function

In order to minimize $E(C_n^q)$, given by expression (41), the first and the second order derivatives with respect to μ_n are given by:

$$\begin{aligned} \frac{\partial}{\partial\mu_n}E(C_n^q) &= 2c_2^q\mu_n + (c_2^q - c_1^q)\left[\sigma_n\phi\left(\frac{\mu_n}{\sigma_n}\right) + \sigma_n\mu_n\frac{\partial}{\partial\mu_n}\phi\left(\frac{\mu_n}{\sigma_n}\right) + \right. \\ &\left. -2\mu_n\Phi\left(-\frac{\mu_n}{\sigma_n}\right) - \left(\mu_n^2+\sigma_n^2\right)\frac{\partial}{\partial\mu_n}\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] = 0 \ , \end{aligned} \tag{42}$$

$$\begin{aligned} \frac{\partial^2}{\partial\mu_n^2}E(C_n^q) &= 2c_2^q + (c_2^q - c_1^q)\left[2\sigma_n\frac{\partial}{\partial\mu_n}\phi\left(\frac{\mu_n}{\sigma_n}\right) + \sigma_n\mu_n\frac{\partial^2}{\partial\mu_n^2}\phi\left(\frac{\mu_n}{\sigma_n}\right) + \right. \\ &\left. -2\Phi\left(-\frac{\mu_n}{\sigma_n}\right) - 4\mu_n\frac{\partial}{\partial\mu_n}\Phi\left(-\frac{\mu_n}{\sigma_n}\right) - (\mu_n^2+\sigma_n^2)\frac{\partial^2}{\partial\mu_n^2}\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] \ . \end{aligned} \tag{43}$$

Equations (42) and (43) can be computed from the first and second order derivatives of $\phi\left(\frac{\mu_n}{\sigma_n}\right)$ and $\Phi\left(-\frac{\mu_n}{\sigma_n}\right)$. With respect to $\phi\left(\frac{\mu_n}{\sigma_n}\right)$, given by

$$\phi\left(\frac{\mu_n}{\sigma_n}\right) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{\mu_n^2}{\sigma_n^2}\right) \ ,$$

these are:

$$\begin{aligned} \frac{\partial}{\partial\mu_n}\phi\left(\frac{\mu_n}{\sigma_n}\right) &= \frac{1}{\sqrt{2\pi}}\left(-\frac{\mu_n}{\sigma_n^2}\right)\exp\left(-\frac{1}{2}\frac{\mu_n^2}{\sigma_n^2}\right) = -\frac{\mu_n}{\sigma_n^2}\phi\left(\frac{\mu_n}{\sigma_n}\right) \\ \frac{\partial^2}{\partial\mu_n^2}\phi\left(\frac{\mu_n}{\sigma_n}\right) &= -\frac{1}{\sigma_n^2}\phi\left(\frac{\mu_n}{\sigma_n}\right) + \frac{\mu_n^2}{\sigma_n^4}\phi\left(\frac{\mu_n}{\sigma_n}\right) \ . \end{aligned}$$

While considering $\Phi(-\frac{\mu_n}{\sigma_n})$, given by:

$$\Phi(-\frac{\mu_n}{\sigma_n}) = \int_{-\infty}^{-\frac{\mu_n}{\sigma_n}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{\mu_n^2}{\sigma_n^2}\right) \ ,$$

the derivatives are obtained as follows:

$$\frac{\partial}{\partial \mu_n}\Phi\left(-\frac{\mu_n}{\sigma_n}\right) = -\frac{1}{\sigma_n}\phi\left(-\frac{\mu_n}{\sigma_n}\right) \quad \text{and} \quad \frac{\partial^2}{\partial \mu_n^2}\Phi\left(-\frac{\mu_n}{\sigma_n}\right) = \frac{\mu_n}{\sigma_n^3}\phi\left(-\frac{\mu_n}{\sigma_n}\right) \ .$$

Therefore, the first derivative of the expected costs is:

$$\begin{aligned} \frac{\partial}{\partial \mu_n}E(C_n^q) &= 2c_2^q\mu_n + (c_2^q - c_1^q)\left[\sigma_n\phi\left(\frac{\mu_n}{\sigma_n}\right) - \frac{\mu_n^2}{\sigma_n}\phi\left(\frac{\mu_n}{\sigma_n}\right) + \right. \\ &\quad \left. -2\mu_n\Phi\left(-\frac{\mu_n}{\sigma_n}\right) + \frac{\mu_n^2}{\sigma_n}\phi\left(\frac{\mu_n}{\sigma_n}\right) + \sigma_n\phi\left(\frac{\mu_n}{\sigma_n}\right)\right] \\ &= 2c_2^q\mu_n + 2(c_2^q - c_1^q)\left[\sigma_n\phi\left(\frac{\mu_n}{\sigma_n}\right) - \mu_n\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] \ , \end{aligned} \tag{44}$$

while the second order optimality condition is given by:

$$\begin{aligned} \frac{\partial^2}{\partial \mu_n^2}E(C_n^q) &= 2c_2^q + (c_2^q - c_1^q)\left[-2\frac{\mu_n}{\sigma_n}\phi\left(\frac{\mu_n}{\sigma_n}\right) - \frac{\mu_n}{\sigma_n}\phi\left(\frac{\mu_n}{\sigma_n}\right) + \frac{\mu_n^3}{\sigma_n^3}\phi\left(\frac{\mu_n}{\sigma_n}\right) + \right. \\ &\quad \left. -2\Phi\left(-\frac{\mu_n}{\sigma_n}\right) + \frac{4\mu_n}{\sigma_n}\phi\left(\frac{\mu_n}{\sigma_n}\right) - \frac{\mu_n^3}{\sigma_n^3}\phi\left(\frac{\mu_n}{\sigma_n}\right) - \frac{\mu_n}{\sigma_n}\phi\left(\frac{\mu_n}{\sigma_n}\right)\right] \\ &= 2c_2^q + (c_2^q - c_1^q)\left[-2\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] \\ &= 2c_2^q\left[1 - \Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] + 2c_1^q\left[\Phi\left(-\frac{\mu_n}{\sigma_n}\right)\right] > 0 \ . \end{aligned} \tag{45}$$