Exchange Algorithms for Constructing Model-Robust Experimental Designs

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Abstract

Optimal experimental design procedures, utilizing criteria such as $\mathcal{D}$-optimality, are useful for producing experimental designs for quantitative responses, often under nonstandard conditions such as constrained design spaces. However, these methods require *a priori* knowledge of the exact form of the response function, an often unrealistic assumption. Model-robust designs are those which, from our perspective, are efficient with respect to a set of possible models. In this paper, we develop a model-robust technique which, when the possible models are nested, is $\mathcal{D}$-optimal with respect to an associated multiresponse model. In addition to providing a justification for the procedure, this motivates the generalization of a modified Fedorov exchange algorithm, which is developed and used to construct exact model-robust designs. We give several examples and compare our designs with two model-robust procedures in the literature.

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Introduction and Motivation

Since Kiefer (1959) debuted the idea of optimal design of experiments, a vast literature has grown up around the notion of choosing a design based upon some numerical criterion. The most common is $\mathcal{D}$-optimality, which chooses the design minimizing the generalized variance of the regression parameter estimates. Though standard designs can be used in most design situations, optimal procedures are useful when, for instance, there are constraints on the design space or some factors are categorical. However, optimal design procedures have been criticized (Box and Draper 1959) because they require complete knowledge of the form of the regression function, though this knowledge is rarely at hand. Subsequently, techniques have been developed which produce designs that are in some way robust to departures from the assumed model.

For instance, optimal designs are often used in mixture experiments because of the constrained nature of the design region. Heinsman and Montgomery (1995) describe an experiment involving a household factor with four surfactant mixture factors. Beyond the mixture constraint, the factors were restricted as well which made an optimal design natural. However, such a design would require the complete specification of the form of the mixture regression model. For instance a special cubic Scheffé polynomial model might be chosen, though it is unknown before the experiment whether this is the correct model. We provide a procedure which allows the experimenter to obtain a design which does not assume a single model form, but rather accounts for a class of user-specified models. We revisit this example later.

Model-robustness has enjoyed significant development over the years, primarily in the hands of theoreticians whose work has provided insights into specific problems and the tradeoff between bias and variance in the assessment of optimal designs; see, for instance Montepiedra and Fedorov (1997), Dette and Franke (2001), Fang and Wiens (2003), Zhou

(2008). On a practical level, much of this work lacks an intuitive framework within which an experimenter might work. In fact, as Chang and Notz (1996) admit in a review of the research literature just mentioned, these model-robust methods have more value as perspicacious descriptions which warn of the dangers of ignoring the issue than as useful prescriptions which would allow their adoption by practitioners.

Even further, nearly all of this research employs Kiefer's continuous design theory which, while mathematically elegant and tractable, produces designs optimal for asymptotically large sample sizes. In contrast, most applications in the physical sciences and engineering require optimal designs for a relatively small number of runs, i.e. discrete, or exact, designs. Consequently, commercial software implementations employ exchange algorithms for fixed sample sizes, including the Fedorov exchange algorithm (Fedorov 1972), DETMAX (Mitchell 1974), and the k-exchange algorithm (Johnson and Nachtsheim 1983).

There is remarkably little work done in accessible discrete methods for model-robust designs. A mean squared error criterion reminiscent of Box and Draper (1959) was proposed by Welch (1983), along with an accompanying DETMAX-like exchange algorithm. DuMouchel and Jones (1994) use a Bayesian approach to provide some protection against specified terms not in the assumed model, but their method requires specification of a prior precision parameter and does not explicitly guard against more than two models; i.e. the assumed model and one that includes the potential terms. Still, this approach formalizes the *ad hoc* practice of adding center points to test for lack of fit and has spawned significant follow-up work, such as Neff (1996), Goos et al. (2005), and Jones et al. (2008). Heredia-Langner et al. (2004) allow protection against multiple models by utilizing a desirability function to incorporate information about each possible model. The necessary optimization is performed by a genetic algorithm, which introduces additional complexity in implementation.

We propose a new, practical method which produces designs robust for a set of user-defined possible models. These ideas are motivated by the a connection between multiresponse regression (Zellner 1962), multiresponse optimal design (Fedorov 1972), and a continuous model-robust optimal design technique due to Läuter (1974). To implement these ideas, we develop multiresponse exchange algorithms which generalize existing univariate methods. As far as we know, only Huizenga et al. (2002) has generalized the basic exchange algorithm of Fedorov (1972) to the multiresponse case, although it is not used to construct model-robust designs.

The paper is organized as follows. In the next section we give the technical background and describe the basic approach taken to find model-robust designs. We then review some basic univariate exchange algorithms and give a multiresponse determinant-updating formula, a simplification of which is used to drive a multiresponse and/or model-robust exchange algorithm. We next give several examples illustrating our method and compare our designs to those of DuMouchel and Jones (1994) and Heredia-Langner et al. (2004). We conclude with discussion of the procedure and its results.

Setting and Proposed Approach

Suppose one is interested in performing an experiment with a single quantitative response variable, y, and a factors (quantitative or categorical), $\mathbf{x} = (x_1, \ldots, x_a)$. We assume that the classical univariate linear regression model will be fit, where $y_i = f'(\mathbf{x}_i)\boldsymbol{\beta} + \epsilon_i, i = 1, \ldots, n$ with $\boldsymbol{\beta}$ a p-vector of parameters and $f(\mathbf{x})$ the p-vector valued model function, though p and the precise form of $f(\mathbf{x})$ are unknown. In matrix notation, we have $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ where $\mathbf{y}$ is an n-vector, $\mathbf{X}$ is an $n \times p$ expanded design matrix, and $\boldsymbol{\epsilon}$ is also an n-vector with $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $Var(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}_n$. We assume also that the least squares criterion is used to estimate $\boldsymbol{\beta}$, in which case the estimator is $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ with $Var(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X}'\mathbf{X})^{-1}$.

To fit such a model, the design must be chosen and y_i observed at each of the designs points, $\mathbf{x}_i$. Let χ be the design space, Ξ be the set of all possible designs and $\xi_n(\mathbf{x}) \in \Xi$ be a discrete, n-point design:

$$\xi_n = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \ldots & \mathbf{x}_d \\ n_1 & n_2 & \ldots & n_d \end{pmatrix} \tag{1}$$

where n is the total number of experiments, and n_i, $i, \ldots, d$ is the number of experiments performed at design point, $\mathbf{x}_i$. We define the information matrix in this case as $\mathbf{M} = \sigma^{-2} \sum_{i=1}^{n} f(\mathbf{x}) f'(\mathbf{x})$ and in the specific instance of the linear regression model, $\mathbf{M} = (\mathbf{X}'\mathbf{X})/\sigma^2 = [Var(\hat{\boldsymbol{\beta}})]^{-1}$.

An optimal design approach would attempt to find the n points, $\mathbf{x}_i \in \chi, i = 1, \ldots, n$, such that some criterion, $\phi(\mathbf{M})$, is optimized. Many criteria have been proposed, but probably the most popular and mathematically tractable is the $\mathcal{D}$-optimality, for which $\phi(\mathbf{M}) = |\mathbf{M}|$. Such an optimal design minimizes the volume of the confidence ellipsoid of the parameters.

Since the precise form of $f(\mathbf{x})$ is generally not known, we might make the weaker assumption that there exists a set of r possible models $\mathcal{F}$ that might be fit. Läuter (1974) presented this idea for continuous designs ξ, and introduced a model-robust criterion similar to $\phi\left(\mathbf{M}_{\mathcal{F}}(\xi)\right) = \prod_{f \in \mathcal{F}} |\mathbf{M}_f(\xi)|$, where $\mathbf{M}_{\mathcal{F}} = (\mathbf{M}_1, \ldots, \mathbf{M}_r)$ and $\mathbf{M}_f$ is the information matrix for model f. Thus, the design which maximizes $\phi(\mathbf{M}_{\mathcal{F}}(\xi))$ over all possible designs might be considered robust to the models in $\mathcal{F}$. Cook and Nachtsheim (1982) utilized this idea to develop linear-optimal designs focusing on prediction. Later, Dette (1990) used the theory of canonical moments to give more explicit solutions for this product criterion. These papers, however, are limited to continuous designs and unconstrained cuboidal design regions.

Our discrete approach springs from Laüter's idea, since allowing the experimenter to define a class of possible models is practically compelling. When model-robustness is viewed in this way, it is closely related to multiresponse optimal design, which has a literature in its own right; see Fedorov (1972), Khuri and Cornell (1987), Chang (1997), and Atashgah and Seifi (2009). These methods are based upon a multiresponse regression model due to Zellner (1962) which allows the functional form of the factors to be different for each response and can produce more precise estimates of the regression parameters by considering the covariance structure of the responses.

Zellner's seemingly unrelated regression (SUR) model, with r responses, can be written as

$$\begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_r \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{X}_r \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \\ \vdots \\ \boldsymbol{\beta}_r \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon}_1 \\ \boldsymbol{\epsilon}_2 \\ \vdots \\ \boldsymbol{\epsilon}_r \end{pmatrix} \tag{2}$$

where each $\mathbf{y}_i$ and $\boldsymbol{\epsilon}_i$ are n-vectors, $\boldsymbol{\beta}_i$ is a q_i-vector, and $\mathbf{X}_i$ is a $n \times q_i$ expanded design matrix for response i and the total number of parameters is $\sum_{i=1}^{r} q_i = q$. It is assumed that the n observations are independent, but the r responses for the i^{th} observation are correlated as specified by the $r \times r$ covariance matrix $\mathbf{\Sigma}$. This leads to an error covariance matrix which is $\mathbf{\Omega} = \mathbf{\Sigma} \otimes \mathbf{I}_n$ where '$\otimes$' is the Kronecker product. Consequently, the generalized least squares estimator is $\hat{\boldsymbol{\beta}}^* = (\mathbf{Z}'\mathbf{\Omega}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{\Omega}^{-1}\mathbf{Y}$ with $Var(\hat{\boldsymbol{\beta}}^*) = (\mathbf{Z}'\mathbf{\Omega}^{-1}\mathbf{Z})^{-1}$ where

$$\mathbf{Z} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{X}_r \end{pmatrix} \tag{3}$$

as seen in (2). Then the $q \times q$ multiresponse information matrix is $\mathbf{M}_m = \mathbf{Z}'\mathbf{\Omega}^{-1}\mathbf{Z}$. Thus, for a given $\mathbf{\Sigma}$, to find a multiresponse $\mathcal{D}$-optimal design, one must find that which maximizes $|\mathbf{M}_m|$, which, as in the univariate case, will be the design which minimizes the volume of the confidence ellipsoid for the parameters.

Notice, however, that finding the multiresponse optimal design for r responses with different regression functions should give a design that is simultaneously "good" for all the response models, though not optimal for any particular one. Consequently, when a univariate model-robust design is viewed as one which performs well for a set of specified models, finding such a design is similar to a parallel multiresponse situation in which there are r response models and we calculate the corresponding multiresponse $\mathcal{D}$-optimal design. Recently, we discovered a technical report (Emmett et al. 2007) which makes the same connection, though the basis of our work is independent of theirs.

Results by Bischoff (1993) and Kurotschka and Schwabe (1996) prove that when the models are nested multiresponse optimal designs are invariant to $\boldsymbol{\Sigma}$. Moreover, since our primary concern is model-robustness, it seems reasonable to assume the identity matrix as the covariance between the r "responses" or models, which when the models in $\mathcal{F}$ are nested gives the attractive multiresponse $\mathcal{D}$-optimal interpretation for the model-robust design. To implement these ideas, we will develop a multivariate generalization of the determinant-updating formula used in univariate exchange algorithms, then use a simplification when $\boldsymbol{\Sigma} = \mathbf{I}$ for our model-robust exchange algorithm.

Multiresponse and Model-Robust Exchange Algorithms

In this section we first review the basic univariate exchange algorithms upon which our methods are based. Then we present a generalization to the matrix-updating formulas used in the univariate procedures, as well as a simplification when $\boldsymbol{\Sigma} = \mathbf{I}$. Finally, we introduce our model-robust exchange algorithm, which utilizes this simplification to avoid calculating determinants when evaluating potential exchanges.

Univariate Exchange Algorithms

The first univariate exchange algorithm (Fedorov 1972) considered exchanges between each design point and points in a candidate list, a discretized version of the design space. At each iteration, the exchange was made which most increases the determinant of the information matrix. He exploited a determinant-updating formula to alleviate the considerable computational burden this problem imposed. Specifically, given design ξ_n, he showed that if $\mathbf{x}_j \in \xi_n$ is exchanged for $\mathbf{x} \in \chi$ resulting in the new design $\tilde{\xi}_n$,

$$|\mathbf{M}(\tilde{\xi}_n)| = |\mathbf{M}(\xi_n)|\,(1 + \Delta(\mathbf{x}_j, \mathbf{x}, \xi_n)) \tag{4}$$

where

$$\Delta(\mathbf{x}_j, \mathbf{x}, \xi_n) = \mathbf{V}(\mathbf{x}, \xi_n) - \mathbf{V}(\mathbf{x}, \xi_n)\mathbf{V}(\mathbf{x_i}, \xi_n) + \mathbf{V}^2(\mathbf{x}, \mathbf{x}_j, \xi_n) - \mathbf{V}(\mathbf{x}_j, \xi_n) \tag{5}$$

under the assumption that $\sigma^2 = 1$, with $\mathbf{V}(\mathbf{x}, \xi_n) = f'(\mathbf{x})\mathbf{M}^{-1}(\xi_n)f(\mathbf{x})$ and $\mathbf{V}(\mathbf{x}, \mathbf{x}_j, \xi_n) = f'(\mathbf{x})\mathbf{M}^{-1}(\xi_n)f(\mathbf{x}_j)$. The Fedorov algorithm is as follows:

1. Initialize algorithm: Begin with a nonsingular design; construct grid, $\mathcal{G} \subset \chi$
2. Let $j = 1$.
3. For design point $\mathbf{x}_j$, calculate $\Delta(\mathbf{x}_j, \mathbf{x}, \xi_n)$ as in (5) for all $\mathbf{x} \in \mathcal{G}$. Choose $\mathbf{x}_j^* = \arg\max_{\mathbf{x}\in\chi} \Delta(\mathbf{x}_j, \mathbf{x}, \xi_n)$.
4. Increment j and if $j < n$ return to Step 3. Else choose $j^* = \arg\max_{j\in\{1,\cdots,n\}} \Delta(\mathbf{x}_j, \mathbf{x}_j^*, \xi_n)$ and exchange $\mathbf{x}_{j^*}$ and $\mathbf{x}_{j^*}^*$, updating the determinant.
5. Update the inverse of the information matrix according to the standard rank-2 updating formula (Fedorov 1972)
6. If $\Delta(\mathbf{x}_{j^*}, \mathbf{x}_{j^*}^*, \xi_n) < \epsilon$, STOP. Else return to Step 2.

This algorithm generates a convergent nondecreasing sequence of determinants, but will not in general converge to the global optimum. Therefore, it is necessary to run many instances of the algorithm each with a randomly generated initial design. Despite the cheap determinant updates, the primary drawback to Fedorov's algorithm is its computational demands since N optimizations are required during each iteration.

Cook and Nachtsheim (1980) proposed a modified Fedorov exchange algorithm, which mimics Fedorov's original procedure but exchanges each $\mathbf{x}_j$ and $\mathbf{x}_j^*$ in Step 3. This capitalizes on each of the n optimizations that are performed during each iteration, and seems to be as effective as its archetype. It is actually a special case of the k-exchange algorithm (Johnson and Nachtsheim 1983), which considers only the k least critical design points (those with the smallest prediction variance) for exchange.

In the remainder of this paper, we develop a multiresponse generalization of the modified Fedorov exchange algorithm and use it to construct single response model-robust designs. We focus on this algorithm since we found it to be faster than the original Fedorov algorithm while producing better designs than the k-exchange. Similar extensions to other existing univariate algorithms, such as DETMAX (Mitchell 1974), BLKL (Atkinson et al. 2007), and coordinate-exchange (Meyer and Nachtsheim 1995), could be developed. The latter does not require a candidate list and is computationally attractive, but its sheen is tarnished in the face of multifactor constraints on the design space. Given that computer-generated designs are especially useful when design regions are constrained (e.g. mixture designs), the benefits of a model-robust coordinate-exchange algorithm may be modest.

Model-Robust Exchange Algorithm

Model-robust exchange algorithms arise from a confluence of motivating factors. First, there is a need to develop practical and intuitive tools which allow experimenters to design experiments for nonstandard situations. Since the form of the model is rarely known in advance, traditional optimal design methods fall short in providing the necessary technical machinery.

Secondly, by noting the similarity between multiresponse optimal design and the single response model-robust design problem we might consider the use of existing multiresponse optimal design methods to construct model-robust designs. However, there exists almost no exact design methods for multiresponse optimal design. This has led us to the development of multiresponse optimal design exchange algorithms based on the multiresponse determinant updating formula given in the second Appendix, and using them to produce model-robust designs.

Simplification of Multiresponse Determinant Updating Formula

Recall that q is the total number of parameters in the multiresponse regression model given in (2) and r is the number of responses. In the second Appendix, we give a multiresponse generalization of the determinant updating formula (4), which allows the determinant of the $q \times q$ multiresponse information matrix to be updated by evaluating the determinant of a $2r \times 2r$ matrix when a single point is exchanged.

However, if we assume that $\mathbf{\Sigma} = \mathbf{I}_r$ we can simplify the multiresponse determinant updating formula given in equation (39) (second Appendix) so that the update involves only a scalar. It is well known that the determinant of a block diagonal matrix is the product of the determinants of the blocks. Thus,

$$\begin{aligned} \left|\mathbf{M}_m(\tilde{\xi})\right| = \left|\tilde{\mathbf{Z}}'\tilde{\mathbf{Z}}\right| &= \prod_{i=1}^{m} \left|\tilde{\mathbf{X}}_i{}'\tilde{\mathbf{X}}_i\right| \\ &= \prod_{i=1}^{m} \left|\mathbf{X}_i'\mathbf{X}_i\right| \cdot (1 + \Delta_i(\mathbf{x}_j, \mathbf{x})) \end{aligned} \tag{6}$$

where the last equality follows from the univariate identity (4). This allows us to update the information matrix via (6) instead of (39), which requires a determinant calculation. We are now prepared to describe the proposed model-robust modified Fedorov exchange algorithm.

Model-Robust Modified Fedorov Exchange Algorithm

As in Läuter (1974) we consider, instead of a single model, a finite set of models $\mathcal{F}$ from which the experimenter believes the true model form can be chosen. More specifically, let ξ_n be an n-point design and $\mathbf{M}_i(\xi_n)$ be the information matrix for model i where $f_i \in \mathcal{F}$, $i = 1, \cdots, r$. Suppose that we exchange a design point $\mathbf{x}_j$ for an arbitrary point $\mathbf{x}$ in the design region, resulting in a new design $\tilde{\xi}_n$. Then the model-robust optimization criteria can be written as:

$$\begin{aligned}\boldsymbol{\phi}(\mathbf{M}_{\mathcal{F}}(\tilde{\xi_n})) &= \prod_{i=1}^{r} \left|\mathbf{M}_i(\tilde{\xi_n})\right| \\ &= \prod_{i=1}^{r} |\mathbf{M}_i(\xi_n)| \left(1+\Delta_i(\mathbf{x}_j,\mathbf{x})\right) \\ &= \boldsymbol{\phi}(\mathbf{M}_f(\xi_n)) \prod_{i=1}^{r}(1+\Delta_i(\mathbf{x}_j,\mathbf{x}))\end{aligned} \tag{7}$$

so that for each iteration of the algorithm, we need to just calculate and maximize $\prod_{i=1}^{r}(1+\Delta_i(\mathbf{x}_j,\mathbf{x}))$ where Δ_i is calculated as in (5) for model i. We make a slight adjustment to this criterion so our algorithm will not choose to exchange a point that is so bad that $(1+\Delta_i(\mathbf{x}_j,\mathbf{x})) < 0$ for an even number of models, which would result in a positive value of our criterion even though the exchange is undesirable. Thus, we choose the exchange which maximizes

$$\prod_{i=1}^{r}(1+\Delta_i(\mathbf{x}_j,\mathbf{x}))\mathbb{I}(1+\Delta_i(\mathbf{x}_j,\mathbf{x}) > 0) \tag{8}$$

where $\mathbb{I}$ is the indicator function. By (6) this is equivalent to updating the multiresponse information matrix under the assumption that $\mathbf{\Sigma} = \mathbf{I}$.

Based on the above development, the algorithm is as follows:

1. Initialize algorithm: Begin with a nonsingular design ξ_n; construct grid, $\mathcal{G} \subset \chi$.

2. Let $j = 1$.

3. For design point $\mathbf{x}_j$, calculate (8) for all $\mathbf{x} \in \mathcal{G}$. Choose $\mathbf{x}_j^* = \arg\max_{\mathbf{x}\in\chi} \prod_{i=1}^{r}(1+\Delta_i(\mathbf{x}_j,\mathbf{x}))\mathbb{I}(1+\Delta_i(\mathbf{x}_j,\mathbf{x}) > 0)$.

4. Perform exchange $\mathbf{x}_j^*$ for $\mathbf{x}_j$, updating ξ_n. Update the determinant and also $\left(\mathbf{X}_i'\mathbf{X}_i\right)^{-1}$ for each model using the rank-2 formula in Fedorov (1972).

5. Increment j and if $j < N$ return to Step 3. Else, if $\max_j \prod_{i=1}^{r}(1+\Delta_i(\mathbf{x}_j,\mathbf{x}_j^*)) < 1+\epsilon$, STOP. Else return to Step 2.

As in the univariate algorithm, to find a global optimum for larger problems it is necessary to perform many runs of the algorithm using different initial designs.

Examples

In this section we present several examples illustrating the proposed *model-robust modified Fedorov* (MRMF) exchange algorithm, and compare it with two other exact model-robust design methods in the literature. Before giving the examples, we will briefly describe these methods and discuss how the designs will be evaluated.

DuMouchel and Jones (1994) use a Bayesian approach to provide protection against higher-order terms. They set r terms as primary and s terms as potential and after scaling the two groups to make them nearly orthogonal, they assume an informative prior for the potential terms and calculate a posterior distribution for the parameters with variance $\mathbf{A} = [\mathbf{X}'\mathbf{X} + \mathbf{K}/\tau^2]^{-1}$, where $\mathbf{X} = (\mathbf{X}_{pri}|\mathbf{X}_{pot})$ and $\mathbf{K}$ is a $(r+s) \times (r+s)$ diagonal matrix with 0 on the first r diagonals and 1 on the last s. The prior variance parameter, τ, is to be chosen by the user. Once they have this posterior variance, they simply choose the design that minimizes $|\mathbf{A}|$ using slightly adjusted exchange algorithms.

A distinct advantage of this method is that it can provide protection against models with more parameters than observations. On the other hand, it is not designed to produce model-robust designs with respect to more than two models. Since it is a prominent and rare model-robust technique for exact designs, we compare its results to ours. Difficulties associated with this method are the choice of the prior precision value, $\frac{1}{\tau}$, and how to designate the primary and potential terms. We use $\frac{1}{\tau} = 1$, as recommended by DuMouchel and Jones, but also include designs based upon $\frac{1}{\tau} = 16$. Because of the structure of $\mathbf{A}$, larger prior precision values will result in less consideration of the potential terms as manifested by lower efficiencies for models involving those terms. We also generally assume more primary terms as opposed to less. The results are based upon the implementation of this method in the *SAS*® software's PROC OPTEX (SAS 2004).

Heredia-Langner et al. (2004) used a genetic algorithm to calculate exact model-robust designs. They consider r possible models and use a genetic algorithm to optimize a desirability function which incorporates the determinants of the information matrices of each of the models. Their procedure does not require a candidate list, though implementation of a tuned genetic algorithm is not trivial. Examples 1 and 3 are taken from their paper, which allows comparisons to be made.

We compare designs on the basis of efficiencies with respect to each model $f \in \mathcal{F}$. The $\mathcal{D}$-efficiency for model f is $\mathcal{D}_{eff} = \left(\frac{|\mathbf{M}_f|}{|\mathbf{M}_f^*|}\right)^{1/p}$ where $\mathbf{M}_f^*$ is the information matrix for the design optimal for f alone, and p is the number of parameters for model f. Since determinants can roughly be viewed as measures of volume, this quantity takes the ratio of the volumes and scales the comparison to a per-parameter basis. When the number of parameters is large, the determinants themselves can be orders of magnitude different yet result in a high $\mathcal{D}$-efficiency. Thus, we provide an alternative measure of efficiency which measures the ratio of determinants, which we call $\mathcal{D}$ Volume-efficiency: $\mathcal{DV}_{eff} = \frac{|\mathbf{M}_f|}{|\mathbf{M}_f^*|}$.

For the individual model optimal designs in all examples save the last, Fedorov's algorithm via PROC OPTEX was run 50 times from randomly chosen initial designs and the best final design was chosen. For the final example, the MRMF algorithm was used to find the best designs for the models individually. Furthermore, all model-robust designs produced by the methods in this paper, as well as those based upon DuMouchel and Jones (1994), were also generated based on 50 separate algorithm instances.

Example 1: Constrained Response Surface Experiment

A constrained two-factor example, taken from Heredia-Langner et al. (2004), will serve as an initial example illustrating our method. The design region, shown in 1, is $\chi = \{\mathbf{x} = (x_1, x_2) : -1 \leq x_1, x_2 \leq 1, x_1 + x_2 \leq 1, -0.5 \leq x_1 + x_2\}$, $n = 6$ and the experimenter would like a design robust for a first-order, a first-order with interaction, or full quadratic polynomial; i.e. $\mathcal{F} = \{f_i^{'}(\mathbf{x})\boldsymbol{\beta}_i, 1 \leq i \leq 3, \mathbf{x} \in \chi\}$ where

$$f_1^{'}(\mathbf{x}) = (1, x_1, x_2) \tag{9}$$

$$f_2^{'}(\mathbf{x}) = (1, x_1, x_2, x_1x_2) \tag{10}$$

$$f_3^{'}(\mathbf{x}) = (1, x_1, x_2, x_1x_2, x_1^2, x_2^2) \tag{11}$$

The candidate list for this example consisted of 266 points constituting a grid of resolution 0.1 placed over the design space. For the Bayesian method, we adopt $\frac{1}{\tau} = 1$ and assign $f_{pri}^{'} = (1, x_1, x_2, x_1x_2)$ and $f_{pot}^{'} = (x_1^2, x_2^2)$ in accordance with recommendations in DuMouchel and Jones (1994). We also include in our comparison the model-robust design of Heredia-Langner et al. (2004) as well as the optimal design for the largest model.

The model-robust designs are shown in Figure 1. Three design points are common to all four designs, $\{(0,1),(1,0),(1,-1)\}$, and the MRMF and Bayes methods produced the same design. Table 1 also compares the designs in terms of the determinant, $\mathcal{DV}$-efficiency, and $\mathcal{D}$-efficiency for each of the considered models, and the last column gives the product. The last row gives the determinant of the information matrix for the $\mathcal{D}$-optimal design for each of the models individually, and the efficiencies are calculated using these values.

Even though the Bayesian and MRMF designs seem close to the optimal design for the quadratic model (since their $\mathcal{D}$-efficiency for the quadratic model is nearly 1), the latter produces a poor design with respect to the interaction model. It is also somewhat surprising that the Bayesian method produced the same design as the MRMF method, given that three models were to be guarded against. However, in this simple example the MRMF design for the three models is the same as that obtained when considering only models (10) and (11) and ignoring (9). Therefore, it appears that the first-order model has no effect upon the MRMF algorithm, so that there are essentially two models under consideration, a situation for which the Bayesian procedure is natural.

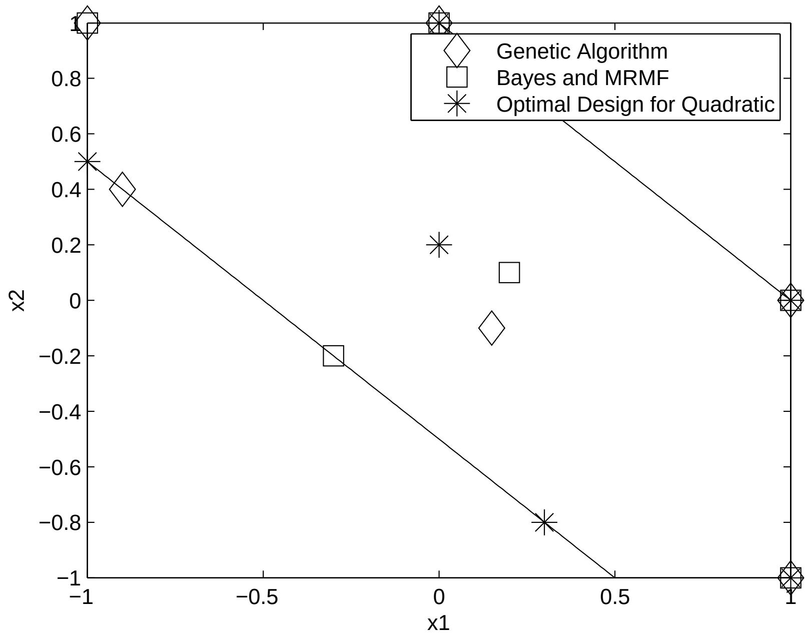

Figure 1: Model-robust Designs for Example 1

Example 2: Hypothetical Constrained 3-factor Experiment

To further explore our method and how it compares to the Bayesian method in particular, consider a three-factor example with design region $\chi = \{\mathbf{x} = (x_1, x_2, x_3) : -1 \leq x_1, x_2, x_3 \leq 1, -1 \leq x_1 + x_2 + x_3 \leq 1, -1 \leq x_1 + x_2 \leq 1, -1 \leq x_1 + x_3 \leq 1, -1 \leq x_2 + x_3 \leq 1\}$ and five models of interest:

$$f_1'(\mathbf{x}) = (1, x_1, x_2, x_3) \tag{12}$$

$$f_2'(\mathbf{x}) = (f_1', x_1x_2, x_1x_3, x_2x_3) \tag{13}$$

$$f_3'(\mathbf{x}) = (f_2', x_1^2, x_2^2, x_3^2) \tag{14}$$

$$f_4'(\mathbf{x}) = (f_3', x_1^2x_2, x_1^2x_3, x_1x_2^2, x_2^2x_3, x_1x_3^2, x_2x_3^2, x_1x_2x_3) \tag{15}$$

$$f_5'(\mathbf{x}) = (f_4', x_1^3, x_2^3, x_3^3) \tag{16}$$

so that $\mathcal{F} = \{f_i'(\mathbf{x})\beta_i, 1 \leq i \leq 5, \mathbf{x} \in \chi\}$.

In particular, assume that the experimenter would like to use $n = 20$ runs and would like

Design	Measure	Model (9)	(10)	(11)	Product
MRMF	Determinant	27.04	33	3.01	2685.88
	$\mathcal{DV}_{eff}$	.531	.677	.968	.348
	$\mathcal{D}_{eff}$	.810	.907	.995	.731
Genetic Algorithm	Determinant	31.14	26.91	2.21	1851.93
	$\mathcal{DV}_{eff}$	.612	.552	.711	.240
	$\mathcal{D}_{eff}$	.849	.862	.945	.692
Bayes ($\frac{1}{\tau} = 1$)	Determinant	27.04	33	3.01	2685.88
	$\mathcal{DV}_{eff}$	.531	.677	.968	.348
	$\mathcal{D}_{eff}$	.810	.907	.995	.731
Optimal Design for (11)	Determinant	31.63	14.35	3.11	1411.60
	$\mathcal{DV}_{eff}$	.622	.294	1	.183
	$\mathcal{D}_{eff}$	.853	.737	1	.629
Optimal (for each model)	Determinant	50.88	48.77	3.11	

Table 1: Determinants, with $\mathcal{DV}$- and $\mathcal{D}$-efficiencies, for Example 1 with $n = 6$, protecting against three models.

a design that can fit each of these models well. To specify the Bayesian procedure, we take as primary all terms in (14) and designate the rest as potential. We give the MRMF design in Table 2, as well as Bayesian designs with $\frac{1}{\tau} = 1$ and $\frac{1}{\tau} = 16$ and the optimal design for the largest model, all using a candidate list consisting of a grid of points with resolution 0.1 placed over the design space.

The Bayesian designs are competitive for most of the models, but the designs lack efficiency for model (15) when compared to the MRMF design, which might be expected since it is in between the primary and full model and as such not explicitly considered. None of the designs perform very well for model (13), though the MRMF design is marginally better. As we expect, when a larger prior precision value is used in the Bayesian procedure, the efficiency of models containing primary terms is reduced, and in this case significantly degrades the design in terms of the product criterion. The optimal design for the largest model is competitive with the Bayesian designs in terms of model-robustness, though the MRMF design would likely be preferred because of its higher efficiencies in models (13), (14), and (15).

Design	Measure	Model (12)	(13)	(14)	(15)	(16)	Product
MRMF	Determinant	6.58e3	5.57e4	1.10e5	3.21e0	5.24e-3	6.78e11
	$\mathcal{DV}_{eff}$	.558	.142	.249	.461	.649	.0059
	$\mathcal{D}_{eff}$	.864	.756	.870	.955	.979	.531
Bayes ($\frac{1}{\tau} = 1$)	Determinant	6.63e3	5.21e4	9.74e4	9.92e-1	7.94e-3	2.65e11
	$\mathcal{DV}_{eff}$	.564	.133	.220	.142	.983	.0023
	$\mathcal{D}_{eff}$	.867	.749	.860	.892	.999	.498
Bayes ($\frac{1}{\tau} = 16$)	Determinant	5.93e3	4.39e4	1.12e5	4.61e-1	4.41e-3	5.93e10
	$\mathcal{DV}_{eff}$	.505	.112	.254	.066	.546	.0005
	$\mathcal{D}_{eff}$	.843	.731	.872	.852	.970	.444
Optimal for (16)	Determinant	6.44e3	4.94e4	9.62e4	7.63e-1	8.07e-3	1.88e11
	$\mathcal{DV}_{eff}$	.548	.126	.218	.110	1	.0017
	$\mathcal{D}_{eff}$	.860	.744	.859	.878	1	.4826
Optimal (for each model)	Determinant	1.18e4	3.93e5	4.42e5	6.97e0	8.07e-3	

Table 2: Determinants, with $\mathcal{D}$-efficiencies and $\mathcal{DV}$-efficiencies, for Example 2 with $n = 20$, protecting against five models.

Example 3: Constrained Mixture Experiment

We now revisit the example (Heinsman and Montgomery 1995) briefly described at the outset. This is a four-factor constrained mixture experiment regarding the formulation of a household product in which 20 runs are available. The design region can be defined thusly:

$$\chi = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4) : \sum_{i=1}^{4} x_i = 1, 0.5 \le x_1 \le 1, 0 \le x_2, x_3 \le 0.5, 0 \le x_4 \le 0.05 \right\} \quad (17)$$

where x_1 is a nonionic surfactant, x_2 is an anionic surfactant, x_3 is a second nonionic surfactant, and x_4 is a zwitterionic surfactant. Because of the dependency induced by the mixture constraint, standard mixture design models are considered which do not include an intercept:

$$f_1'(\mathbf{x}) = (\{x_i, i = 1, \ldots, 4\}) \quad (18)$$

$$f_2'(\mathbf{x}) = (f_1', \{x_i x_j, i < j \le 4\}) \quad (19)$$

$$f_3'(\mathbf{x}) = (f_2', \{x_i x_j x_k, i < j < k \le 4\}) \quad (20)$$

$$f_4'(\mathbf{x}) = (f_3', \{x_i x_j (x_i - x_j), i < j \le 4\}) \quad (21)$$

so that $\mathcal{F} = \{f_i^{'}(\mathbf{x})\beta_i, 1 \leq i \leq 4, \mathbf{x} \in \chi\}$. Heredia-Langner et al. (2004) also used this example, and so we compare our method to their Genetic Algorithm as well as to the Bayesian method of DuMouchel and Jones (1994). For the latter, we use both a standard value for the prior precision, $\frac{1}{\tau} = 1$, and a larger precision value ($\frac{1}{\tau} = 16$), with primary terms all those except those unique to f_4, which are regarded as potential.

Since this is a large mixture design, we supplemented a regular grid (resolution 0.01) with extreme vertices and approximate centroids of the design region using code as described in Piepel (1988).

		Model				
Design	Measure	(18)	(19)	(20)	(21)	Product
MRMF	Determinant	5.31e-2	7.22e-22	2.65e-43	8.36e-78	8.49e-143
	$\mathcal{DV}_{eff}$	.281	.336	.365	.921	.032
	$\mathcal{D}_{eff}$	.728	.897	.931	.996	.606
Genetic Algorithm	Determinant	5.23e-2	7.46e-22	2.90e-43	7.80e-78	8.83e-143
	$\mathcal{DV}_{eff}$	.277	.347	.399	.859	.033
	$\mathcal{D}_{eff}$	.725	.900	.937	.992	.607
Bayes ($\frac{1}{\tau} = 1$)	Determinant	5.46e-2	6.74e-22	2.24e-43	9.08e-78	7.48e-143
	$\mathcal{DV}_{eff}$	.289	.313	.308	1	.028
	$\mathcal{D}_{eff}$	.733	.890	.919	1	.600
Bayes ($\frac{1}{\tau} = 16$)	Determinant	5.64e-2	6.12e-22	3.01e-43	3.08e-78	3.20e-143
	$\mathcal{DV}_{eff}$	.298	.285	.414	.339	.012
	$\mathcal{D}_{eff}$	.739	.882	.939	.947	.580
Optimal Design for (21)	Determinant	5.46e-2	6.74e-22	2.24e-43	9.08e-78	7.48e-143
	$\mathcal{DV}_{eff}$	.289	.313	.308	1	.028
	$\mathcal{D}_{eff}$	.733	.890	.919	1	.600
Optimal (for each model)	Determinant	1.89e-1	2.15e-21	7.26e-43	9.08e-78	

Table 3: Determinant function values, $\mathcal{D}$-efficiencies, and $\mathcal{DV}$-efficiencies for Example 3 with $n = 20$, protecting against four models.

In Table 3, our method can be seen to be competitive with the Genetic Algorithm, though their design is slightly superior by our product optimality criterion. This is likely a function of the discretization in our candidate list. Note that the optimal design for model (21) has a significantly higher objective function value (9.08e-78) than that given in Heredia-Langner et al. (2004), though theirs was asserted to have been obtained from PROC OPTEX in SAS as well.

It is the case again in this example that the best design found by the MRMF method is relatively close to that of the optimal design for the largest model. The Bayesian design with precision of 1 actually chooses the optimal design for the largest model, and shows that this design is competitive with those that look to maximize the product of the determinants. When the precision is increased, we see the same behavior as was noted before: The Bayesian design becomes less efficient for the model that involves potential terms. The resulting Bayesian design gives slightly more balance, but suffers against the product optimality criterion.

Example 4: Mixture Experiment with Disparate Models

For our final example we use an unconstrained mixture experiment by Frisbee and McGinity (1994) with $n = 11$. The response is the glass transition temperature of a certain film with three nonionic surfactant factors. The goal was to minimize this transition temperature, and Frisbee and McGinity fit a traditional polynomial model. However, another class of models, the so-called Becker models (Cornell 1990, Sec. 6.5), were shown by Rajagopal and Castillo (2005) to also fit the data well and lead to a significantly different optimal solution. These models, originally considered to address certain shortcomings in the Sheffé polynomial models, use min$(\cdot)$ instead of prod$(\cdot)$ to model factor interactions.

In this case,

$$\chi = \left\{ \mathbf{x} = (x_1, x_2, x_3) : \sum_{i=1}^{3} x_i = 1, 0 \leq x_i \leq 1, i = 1, 2, 3 \right\} \tag{22}$$

and we take five possible models:

$$f_1'(\mathbf{x}) = (\{x_i, i = 1, 2, 3\}) \tag{23}$$

$$f_2'(\mathbf{x}) = (f_1', \{x_i x_j, i < j \leq 3\}) \tag{24}$$

$$f_3'(\mathbf{x}) = (f_2', \{x_1 x_2 x_3\}) \tag{25}$$

$$f_4'(\mathbf{x}) = (f_1', \{min(x_i, x_j), i < j \leq 3\}) \tag{26}$$

$$f_5'(\mathbf{x}) = (f_4', \{min(x_1, x_2, x_3)\}) \tag{27}$$

so that $\mathcal{F} = \{f_i'(\mathbf{x})\beta_i, 1 \leq i \leq 5, \mathbf{x} \in \chi\}$.

In addition to the three models we are guarding against, we also examine effectiveness of our design with respect to the model fit by Frisbee and McGinity, as well as the most probable model found *a posteriori* by Rajagopal and Castillo:

$$f_{fm}'(\mathbf{x}) = (x_1, x_2, x_3, x_1 x_3, x_2 x_3) \tag{28}$$

$$f_{rc}'(\mathbf{x}) = (x_1, x_2, x_3, min(x_1, x_3), min(x_2, x_3)) \tag{29}$$

For a candidate list, we used a regular grid with resolution 1/12, which because of the regular design region, contained the vertices and centroids of the region.

With the disparate model types, the Bayes procedure, with its primary and potential factors, cannot be easily applied. Instead, we examine the results of the MRMF design and compare it in Table 4 to the design that was actually used. In terms of efficiency, the actual design is much inferior for all models considered, and cannot even estimate the Becker model. This is the case because the original design includes, in addition to two centroid points, three other points on the interior of the simplex design region.

As seen in Table 4, the MRMF design is optimal for models (25), (26), and (27). This is because the optimal designs for these models individually are interchangeable; i.e. the optimal design for one is also optimal for another. Note that since the models are not nested we do not have the multiresponse $\mathcal{D}$-optimality interpretation.

Design	Measure	Model (23)	(24)	(25)	(26)	(27)	(28)	(29)
MRMF	Determinant	19.81	5.91e-3	5.36e-6	0.569	2.78e-2	6.61e-2	1.46
	$\mathcal{DV}_{eff}$	.413	.756	1	1	1	.352	.486
	$\mathcal{D}_{eff}$	.745	.954	1	1	1	.812	.866
Frisbee and McGinity	Determinant	8.25	1.22e-3	1.51e-6	.146	8.82e-3	2.30e-2	.588
	$\mathcal{DV}_{eff}$	.172	.156	.283	.257	.317	.123	.196
	$\mathcal{D}_{eff}$	.556	.733	.835	.797	.849	.658	.722
Optimal (for each model)	Determinant	48	7.8e-3	5.36e-6	.569	2.73e-2	.188	3

Table 4: Objective Function Values, $\mathcal{D}$-efficiencies, and $\mathcal{DV}$-efficiencies for Example 4, $n = 11$, protecting against 5 Models.

Discussion

The Model-robust Modified Fedorov (MRMF) exchange algorithm presented in this paper provides a natural tool with which to find designs when an optimal design is desired but the model-form is unknown. The mechanism to achieve this is intuitive and simple: The experimenter chooses r models for which he/she would like to design. Then, a design is found which maximizes the product of the determinant of the information matrices of each of the models. In the case that the models under consideration are nested, this is the $\mathcal{D}$-optimal design for the associated multiresponse model with r responses and thus minimizes the volume of the confidence ellipsoid of the parameters.

Furthermore, the MRMF method produces designs that are competitive, with simpler algorithmic machinery, than the Genetic Algorithm approach of Heredia-Langner et al. (2004). The strength of the MRMF method with respect to the GA technique is that it is automatic and a straightforward extension of commonly used exchange algorithms. The GA requires tuning of several parameters and is nontrivial to implement effectively.

We also compared our procedure to the Bayesian method of DuMouchel and Jones (1994), a widely available model-robust technique. We initially hypothesized that the Bayesian method would suffer when confronted with multiple possible models, since it categorizes terms into just two groups. This is supported by Example 2, though the procedure performed fairly well in Examples 1 and 3. The choice of $\frac{1}{\tau}$ certainly affects the model-robustness of the design; indeed for certain values of $\frac{1}{\tau}$ (i.e. $\frac{1}{\tau} = 1$ in Example 3) the method seems to produce a design optimal for the highest-order model, while for large enough values of $\frac{1}{\tau}$ the full model is not even estimable. The choice of terms as primary or potential also makes an impact. Our procedure does not suffer from these uncertainties, has a multiresponse $\mathcal{D}$-optimal interpretation (for $\mathcal{F}$ nested) and explicitly considers a larger class of models; it can also handle situations as in Example 4 in which the possible models are disparate and impossible to nest.

One strategy, if faced with a situation necessitating a $\mathcal{D}$-optimal design, might be to design for the highest-order model possible. If, as assumed in this paper, there are a sufficient number of runs to estimate the largest model, one might question whether the efficiency gained in model-robust methods is worth the additional methodology. In certain cases, as in the third example, the gains appear to be limited. But as demonstrated by the first and second examples, significant gains can be made by utilizing the model-robust approach. Therefore, a dedicated procedure based upon accepted univariate exchange algorithms will be useful to produce model-robust designs.

In terms of $\mathcal{D}$-efficiency, the MRMF designs can be seen to favor larger models. In other words, the efficiency of the smaller models suffer as compared to the larger ones. To mitigate this, one might consider the following optimization criterion (Atkinson et al. 2007, Emmett et al. 2007), instead of (7):

$$\boldsymbol{\phi}(\mathbf{M}_{\mathcal{F}}(\tilde{\xi}_n)) = \prod_{i=1}^{r} \left|\mathbf{M}_i(\tilde{\xi}_n)\right|^{1/q_i} \tag{30}$$

where q_i is the number of parameters in the i^{th} model. It is straightforward to derive an exchange algorithm using this criterion–call it the scaled MRMF–which has the effect of shrinking values of dissimilar orders of magnitude toward each other, in essence weighting more heavily those models with fewer parameters. We implemented this procedure using several examples, and the results were surprisingly similar. For instance, for the constrained mixture experiment in Example 3, the scaled MRMF design resulted in a design very close to the MRMF in Table 3. For the hypothetical experiment in Example 2, we observed more of a difference, with $\mathcal{D}$-efficiencies increasing from 86.4% to about 89% for model (12) and from 75.6% to about 78% for model (13), while decreasing the efficiencies of model (14) from 87% to about 85.5% and model (16) from 97.9% to about 96.5%, but still resulting in an unbalanced design in terms of the $\mathcal{D}$-efficiencies.

The model-robust criterion used in this paper could easily be extended to include prior information in terms of model weights, if certain models are preferred over the others. However, since this work was motivated in part by multiresponse optimal design theory, the minimal volume of the parameter confidence ellipsoid interpretation of $\mathcal{D}$-optimality is used and thus we only consider equally weighted models. Furthermore, the relative ineffectiveness of the scaled MRMF to provide designs with balanced $\mathcal{D}$-efficiencies underscores the difficulty in balancing the designs using weights.

Finally, assume that T_e is the time it takes to run the univariate exchange algorithm. The runtime for these model-robust algorithms should be rT_e where r is the number of models considered. Commercial software programs have fast implementations of exchange algorithms, so the computational burden imposed by a similarly implemented model-robust exchange algorithm should not be heavy.

Note: All designs referred to in this paper, as well as Matlab code to generate the MRMF designs in the four examples, are available at `http://www2.ie.psu.edu/Castillo/research/EngineeringStatistics/publications.htm`.

Appendix: Matrix Algebra Results

We provide here a collection of results which are necessary to prove Theorem 4. The first is well-known and presented without proof.

Lemma 1 *Let* $\boldsymbol{\Delta}$ *be a block matrix such that*

$$\boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{\Delta}_{11} & \boldsymbol{\Delta}_{12} \\ \boldsymbol{\Delta}_{21} & \boldsymbol{\Delta}_{22} \end{pmatrix} \tag{31}$$

where $\mathbf{\Delta}_{11}$ *is a* $n \times n$ *nonsingular matrix,* $\mathbf{\Delta}_{12}$ *is a* $n \times k$ *matrix,* $\mathbf{\Delta}_{21}$ *is a* $k \times n$ *matrix, and* $\mathbf{\Delta}_{22}$ *is a* $k \times k$ *nonsingular matrix. Then*

$$|\mathbf{\Delta}| = |\mathbf{\Delta}_{11}||\mathbf{\Delta}_{22} - \mathbf{\Delta}_{21}\mathbf{\Delta}_{11}^{-1}\mathbf{\Delta}_{12}| = |\mathbf{\Delta}_{11} - \mathbf{\Delta}_{12}\mathbf{\Delta}_{22}^{-1}\mathbf{\Delta}_{21}||\mathbf{\Delta}_{22}| \tag{32}$$

The next result is a slight generalization of an identity given in Schott (1997).

Lemma 2 *Let* $\mathbf{M}$ *be* $n \times n$*,* $\mathbf{A}$ *be* $n \times k$ *and* $\mathbf{B}$ *be* $k \times n$*. Then*

$$|\mathbf{M} + \mathbf{AB}| = \begin{vmatrix} \mathbf{M} & \mathbf{A} \\ -\mathbf{B} & \mathbf{I}_k \end{vmatrix} \tag{33}$$

Proof. Using basic matrix multiplication, it is true that

$$\begin{pmatrix} \mathbf{M} & \mathbf{A} \\ -\mathbf{B} & \mathbf{I}_k \end{pmatrix} \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{B} & \mathbf{I}_k \end{pmatrix} = \begin{pmatrix} \mathbf{M} + \mathbf{AB} & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_k \end{pmatrix} \tag{34}$$

Taking the determinant of both sides, and using the well known property that the determinant of a product of two matrices is equal to the product of the determinants of the matrices, gives

$$\begin{vmatrix} \mathbf{M} & \mathbf{A} \\ -\mathbf{B} & \mathbf{I}_k \end{vmatrix} \begin{vmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{B} & \mathbf{I}_k \end{vmatrix} = \begin{vmatrix} \mathbf{M} + \mathbf{AB} & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_k \end{vmatrix} \tag{35}$$

and by Lemma 1,

$$\begin{vmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{B} & \mathbf{I}_k \end{vmatrix} = |\mathbf{I}_n||\mathbf{I}_k - \mathbf{0}| = 1 \tag{36}$$

Thus,

$$\begin{aligned}\begin{vmatrix}\mathbf{M} & \mathbf{A}\\ -\mathbf{B} & \mathbf{I}_k\end{vmatrix} &= \begin{vmatrix}\mathbf{M}+\mathbf{AB} & \mathbf{A}\\ \mathbf{0} & \mathbf{I}_k\end{vmatrix}\\ &= |\mathbf{I}_n||\mathbf{M}+\mathbf{AB}-\mathbf{AI}_k\mathbf{0}|\\ &= |\mathbf{M}+\mathbf{AB}|\end{aligned}$$

where the second equality follows from another appeal to Lemma 1. ∎

The final determinant lemma simply combines the two previous results and is a slight generalization of Lemma 2.5.1 in Fedorov (1972).

Lemma 3 *Let* $\mathbf{M}$ *be a nonsingular* $n \times n$ *matrix, let* $\mathbf{A}$ *be a* $n \times k$ *matrix and let* $\mathbf{B}$ *be an* $k \times n$ *matrix; then*

$$|\mathbf{M}+\mathbf{AB}| = |\mathbf{M}||\mathbf{I}_k + \mathbf{BM}^{-1}\mathbf{A}| \tag{37}$$

Proof. Lemma 2 gives that

$$|\mathbf{M}+\mathbf{AB}| = \begin{vmatrix}\mathbf{M} & \mathbf{A}\\ -\mathbf{B} & \mathbf{I}_k\end{vmatrix} \tag{38}$$

and then by Lemma 1 we get what we wanted to prove. ∎

Appendix: Multiresponse Determinant Updating Formula

We can prove a multivariate generalization of a result (Fedorov 1972, Lemma 3.2.1) from which (4) is derived, using the same sorts of arguments. This is essentially identical to the result given in Huizenga et al. (2002), but we present it here with an explicit proof.

Theorem 4 *Let* ξ_n *be an exact design consisting of points* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *and* $\tilde{\xi}_n$ *be the design produced when* $\mathbf{x}_{j_1}, \mathbf{x}_{j_2}, \ldots, \mathbf{x}_{j_\ell}$, $\mathbf{x}_{j_i} \in \xi_n$, *are exchanged for* $\tilde{\mathbf{x}}_k \in \chi$, $k = 1, \ldots, \ell$. *Further, let* $\mathbf{M}_m(\xi_n)$ *be the* $q \times q$ *multivariate information matrix of the design* ξ_n *and*

$\boldsymbol{\gamma}(\mathbf{x}_{j_k})$ be the $r \times q$ multiresponse basis matrix, where $q = p_1 + ... + p_r$ and p_i is the number of parameters for the i^{th} response; then

$$|\mathbf{M}_m(\tilde{\xi_n})| = |\mathbf{M}_m(\xi_n)||\mathbf{I}_{2\ell r} + \mathbf{A}_2^{'}\mathbf{M}_m^{-1}(\xi_n)\mathbf{A}_1| \tag{39}$$

where

$$\mathbf{A}_1 = \left(-\boldsymbol{\gamma}'(\mathbf{x}_{j_1})\boldsymbol{\Sigma}^{-1/2}, \boldsymbol{\gamma}'(\tilde{\mathbf{x}}_1)\boldsymbol{\Sigma}^{-1/2}, \ldots, -\boldsymbol{\gamma}'(\mathbf{x}_{j_\ell})\boldsymbol{\Sigma}^{-1/2}, \boldsymbol{\gamma}'(\tilde{\mathbf{x}}_\ell)\boldsymbol{\Sigma}^{-1/2}\right) \tag{40}$$

and

$$\mathbf{A}_2 = \left(\boldsymbol{\gamma}'(\mathbf{x}_{j_1})\boldsymbol{\Sigma}^{-1/2}, \boldsymbol{\gamma}'(\tilde{\mathbf{x}}_1)\boldsymbol{\Sigma}^{-1/2}, \ldots, \boldsymbol{\gamma}'(\mathbf{x}_{j_\ell})\boldsymbol{\Sigma}^{-1/2}, \boldsymbol{\gamma}'(\tilde{\mathbf{x}}_\ell)\boldsymbol{\Sigma}^{-1/2}\right) \tag{41}$$

and both matrices are $q \times 2\ell r$.

Proof. By definition,

$$\mathbf{M}(\tilde{\xi_n}) = \mathbf{M}(\xi_n) - \sum_{k=1}^{\ell} \boldsymbol{\gamma}'(\mathbf{x}_{j_k})\Sigma^{-1}\boldsymbol{\gamma}(\mathbf{x}_{j_k}) + \sum_{k=1}^{\ell} \boldsymbol{\gamma}'(\tilde{\mathbf{x}}_k)\Sigma^{-1}\boldsymbol{\gamma}(\tilde{\mathbf{x}}_k) \tag{42}$$

Now,

$$\mathbf{A}_1\mathbf{A}_2^{'} = \left(-\boldsymbol{\gamma}'(\mathbf{x}_{j_1})\boldsymbol{\Sigma}^{-1/2}, \boldsymbol{\gamma}'(\tilde{\mathbf{x}}_1)\boldsymbol{\Sigma}^{-1/2}, \ldots, -\boldsymbol{\gamma}'(\mathbf{x}_{j_\ell})\boldsymbol{\Sigma}^{-1/2}, \boldsymbol{\gamma}'(\tilde{\mathbf{x}}_\ell)\boldsymbol{\Sigma}^{-1/2}\right) \begin{pmatrix} \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\gamma}(\mathbf{x}_{j_1}) \\ \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\gamma}(\tilde{\mathbf{x}}_1) \\ \vdots \\ \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\gamma}(\mathbf{x}_{j_\ell}) \\ \boldsymbol{\Sigma}^{-1/2}\boldsymbol{\gamma}(\tilde{\mathbf{x}}_\ell) \end{pmatrix} \tag{43}$$

$$= \sum_{k=1}^{\ell} -\boldsymbol{\gamma}'(\mathbf{x}_{j_k})\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}(\mathbf{x}_{j_k}) + \boldsymbol{\gamma}'(\tilde{\mathbf{x}}_k)\boldsymbol{\Sigma}^{-1}\boldsymbol{\gamma}(\tilde{\mathbf{x}}_k) \tag{44}$$

This implies that $\mathbf{M}(\tilde{\xi_n}) = \mathbf{M}(\xi_n) + \mathbf{A}_1\mathbf{A}_2^{'}$ and by Lemma 3,

$$|\mathbf{M}(\xi_n) + \mathbf{A}_1\mathbf{A}_2^{'}| = |\mathbf{M}(\xi_n)||\mathbf{I}_{2\ell r} + \mathbf{A}_2^{'}\mathbf{M}^{-1}(\xi_n)\mathbf{A}_1| \tag{45}$$

which implies what we wanted to prove. ∎

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References

Atashgah, A. B. and Seifi, A. (2009). Optimal design of multi-response experiments using semi-definite programming. *Optimization and Engineering*, 10(1):75–90.

Atkinson, A. C., Donev, A. N., and Tobias, R. D. (2007). *Optimum Experimental Designs, with SAS*. Oxford University Press.

Bischoff, W. (1993). On d-optimal designs for linear models under correlated observations with an application to a linear model with multiple response. *Journal of Statistical Planning and Inference*, 37:69–80.

Box, G. E. P. and Draper, N. R. (1959). A basis for the selection of a response surface design. *Journal of the American Statistical Association*, 54(287):622–654.

Chang, S. I. (1997). An algorithm to generate near d-optimal designs for multiple-response surface models. *IIE Transactions*, 29:1073–1081.

Chang, Y.-J. and Notz, W. I. (1996). *Handbook of Statistics, Vol. 13*, chapter Model Robust Designs, pages 1055–1098. Elsevier Science B. V.

Cook, R. D. and Nachtsheim, C. J. (1980). A comparison of algorithms for constructing exact d-optimal designs. *Technometrics*, 22:315–324.

Cook, R. D. and Nachtsheim, C. J. (1982). Model-robust, linear-optimal designs. *Technometrics*, 24:49–54.

Cornell, J. (1990). *Experiments with Mixtures: Designs, Models, and the Analysis of Mixture Data*. Wiley-Interscience, 2nd edition.

Dette, H. (1990). A generalization of $\mathcal{D}$- and $\mathcal{D}_1$-optimal design in polynomial regression. *Annals of Statistics*, 18:1784–1804.

Dette, H. and Franke, T. (2001). Robust designs for polynomial regression by maximizing a minimum of $\mathcal{D}$- and $\mathcal{D}_1$-efficiencies. *The Annals of Statistics*, 29(4):1024–1049.

DuMouchel, W. and Jones, B. (1994). A simple bayesian modification of d-optimal designs to reduce dependence on an assumed model. *Technometrics*, 36(1):37–47.

Emmett, M., Goos, P., and Stillman, E. (2007). D-optimal designs for multiple polynomial responses. Technical report, University of Sheffield.

Fang, Z. and Wiens, D. P. (2003). Robust regression designs for approximate polynomial models. *Journal of Statistical Planning and Inference*, 117:305–321.

Fedorov, V. V. (1972). *Theory of Optimal Experiments*. Academic Press, New York, NY.

Frisbee, S. E. and McGinity, J. W. (1994). Influence of nonionic surfactants on the physical and chemical properties of a biodegradable pseudolatex. *European Journal of Pharmaceutics and Biopharmaceutics*, 40:355–363.

Goos, P., Kobilinsky, A., O'Brien, T. E., and Vandebroek, M. (2005). Model-robust and model-sensitive designs. *Computational Statistics & Data Analysis*, 49:201–216.

Heinsman, J. A. and Montgomery, D. C. (1995). Optimization of a household product formulation using a mixture experiment. *Quality Engineering*, 7:583–600.

Heredia-Langner, A., Montgomery, D. C., Carlyle, W. M., and Borror, C. M. (2004). Model-robust optimal designs: A genetic algorithm approach. *Journal of Quality Technology*, 36(3):263–279.

Huizenga, H. M., Heslenfeld, D. J., and Molenaar, P. C. M. (2002). Optimal measurement conditions for spatiotemporal eeg/meg source analysis. *Psychometrika*, 67(2):299–313.

Johnson, M. E. and Nachtsheim, C. J. (1983). Some guidelines for constructing exact d-optimal designs on convex design spaces. *Technometrics*, 25:271–277.

Jones, B., Lin, D. K. J., and Nachtsheim, C. J. (2008). Bayesian d-optimal supersaturated designs. *Journal of Statistical Planning and Inference*, 138:86–92.

Khuri, A. I. and Cornell, J. A. (1987). *Response Surfaces: Designs and Analyses*. M. Dekker.

Kiefer, J. (1959). Optimum experimental designs. *Journal of the Royal Statistical Society, Series B*, 21:272–304.

Kurotschka, V. G. and Schwabe, R. (1996). *Madan Puri Festschrift*, chapter The Reduction of Design Problems for Multivariate Experiments to Univariate Possibilities and their Limitations, pages 193–204. VSP.

Läuter, E. (1974). Experimental design in a class of models. *Statistics*, 5:379–398.

Meyer, R. K. and Nachtsheim, C. J. (1995). The coordinate-exchange algorithm for constructing exact optimal experimental designs. *Technometrics*, 37(1):60–69.

Mitchell, T. J. (1974). An algorithm for the construction of d-optimal experimental designs. *Technometrics*, 16:203–210.

Montepiedra, G. and Fedorov, V. V. (1997). Minimum bias designs with constraints. *Journal of Statistical Planning and Inference*, 63:97–111.

Neff, A. (1996). *Bayesian two-stage designs under model uncertainty.* PhD thesis, Virginia Polytechnic Institute and State University.

Piepel, G. F. (1988). Programs for generating extreme vertices and centroids of linearly constrained experimental regions. *Journal of Quality Technology*, 20:2:125–139.

Rajagopal, R. and Castillo, E. D. (2005). Model-robust process optimization using bayesian model averaging. *Technometrics*, 47(2):152–163.

SAS (2000-2004). SAS Institute Inc., Version 9.1.3, Cary, NC, USA.

Schott, J. R. (1997). *Matrix Analysis for Statistics.* John Wiley & Sons, Inc.

Welch, W. J. (1983). A mean squared error criterion for the design of experiments. *Biometrika*, 70(1):205–213.

Zellner, A. (1962). An efficient method of estimating seemingly unrelated regressions and tests for aggregation bias. *Journal of the American Statistical Association*, 57:348–368.

Zhou, J. (2008). D-optimal minimax regression designs on discrete design space. *Journal of Statistical Planning and Inference*, 138:4081–4092.