Summary for Lectures 6-7

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We can change coordinates in two variable integrals as we do in the one-dimensional case. If we change to $u = u(x, y)$ and $v = v(x, y)$, the area of a small rectangle approximately changes by (absolute value of) the Jacobian determinant

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}.$$

Thus, the area elements are related by

$$du\ dv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\ dy.$$

Then, you need to determine the bounds of the integral by seeing how the region R or more specifically the boundary of the region R changes under the change of coordinates.

Triple integrals are very similar to double integrals. These integrals are evaluated on a function $f(x, y, z)$ of three variables over a region R in space. In this case, there are six choices for the order of integration with the volume element $dV = dx\,dy\,dz$. To set up the bounds, we look at the total change in the value that is furthest to the right, look at the change in the variable second from the right while the one furthest to the right is fixed, and finally look at the change of the leftmost variable while the other two are fixed or we can look at the change in the first variable when the other two is fixed then look at the shadow in the plane spanned by the other two reducing to a two variable setup. We can also make a change of coordinates to cylindrical coordinates with $x = r\cos\theta, y = r\sin\theta$, and $z = z$ that is directly analogous to polar coordinates. Under this change, the volume element becomes

$$dx\,dy\,dz = rdr\,d\theta\,dz.$$

The applications are also similar. For instance, the volume of a region is the integral of the function $f(x, y, z) = 1$ over the region and the mass is given by the integral of a mass density function $\rho(x, y, z)$. The average value of a function is

$$\bar{f} = \frac{1}{\text{vol}} \iiint_R f(x,y,z)\ dV$$

and the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{1}{\text{mass}} \iiint_R x\rho\ dV$$

with similar formulas for the other variables.

Sections 15.10 and 15.7 in Calculus, 7th Edition, by James Stewart