Summary for Lectures 10-11

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For the remainder of this class, we will be studying the calculus of vector fields. A vector field $\vec{F}$ is a function that assigns a vector to each point. More precisely, a vector field in the plane (or on a region of the plane) assigns a 2-dimensional vector to each point and can be written as

$$\vec{F} = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}.$$

A vector field in space (or on a region of space) assigns a 3-dimensional to each point and can be written as

$$\vec{F} = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}.$$

We have already seen an important example of a vector field: the gradient ∇f of a function is vector field. Not every vector field $\vec{F}$ is the gradient of a function. If it is, we say that $\vec{F} = \nabla f$ is conservative and f is a potential function for $\vec{F}$. A related topic is integrating along a curve (line integrals). We want to integrate a function on the plane or in space along some curve C. To do this, we need to integrate with respect to the arclength ds along the curve. To compute the integral, we need to parametrize the curve. Then, $ds = |\vec{v}(t)|dt$. For instance, the integral is given by

$$\int_C f(x,y)\ ds = \int_{t_1}^{t_2} f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

in the plane (there is a similar formula in space). Using this, we can do many of the same applications of integration as before (mass, average value, ...).

In addition to integrating functions along curves, we would like to integrate vector fields along curves. These integrals are written as $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is a vector field. There are three ways to compute this integral. First, we can take a parameterization $\vec{r}(t)$ of C for $t_1 \le t \le t_2$ so that $d\vec{r} = \frac{d\vec{r}}{dt}\ dt$ and the integral becomes

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \left(\vec{F} \cdot \frac{d\vec{r}}{dt}\right)\ dt.$$

This integral is really just the integral of the function $\vec{F} \cdot \hat{T}$ where $\hat{T}$ is the unit tangent vector to the curve so we also have

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T}\ ds.$$

Finally, we can write $d\hat{r} = \langle dx, dy\rangle$ and $\vec{F} = \langle P, Q\rangle$ so that $F \cdot d\vec{r} = Pdx + Qdy$ and

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\ dx + Qdy.$$

All these formulas generalize in the natural way to higher dimensions. The motivation for this integral is that if $\vec{F}$ is a vector describing a force, then this integral is equal to the work done by this force when moving along the curve C.

Sections 16.1 and 16.2 in Calculus, 7th Edition, by James Stewart