Summary for Lectures 12-14

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We have learned that there is a fundamental theorem of line integrals reminiscent of the fundamental theorem of calculus. The statement of the theorem is that if C is a curve between (x_0, y_0) and (x_1, y_1) then

$$\int_C \nabla f \cdot d\vec{r} = f(x_1, y_1) - f(x_0, y_0)$$

and similarly in higher dimensions. Using this result and a little more work, we get the following theorem.

Theorem. The following three statements are equivalent.
1. $\vec{F}$ is conservative.
2. $\int_C \vec{F} \cdot d\vec{r}$ depends only on the endpoints of C.
3. $\int_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C.

This is a nice result, but we would like to know an easier way to test if $\vec{F}$ is conservative. The following gives us exactly.

Theorem. If $\vec{F} = \langle P, Q \rangle$ is defined and continuously differentiable on a simply connected (no holes, e.g., the whole plane) subset of the plane, then $\vec{F}$ is conservative if and only if $P_y = Q_x$.

Suppose that we know a vector field $\vec{F} = \langle P, Q \rangle$ is conservative. We can find a potential f for $\vec{F}$ by solving $f_x = P$ and $f_y = Q$. This is done by integrating first P with respect to x. This will give $f(x, y) = h(x, y) + g(y)$ for some known function h and an unknown function g since the constant of integration now depends on y. Then, $Q = f_y = h_y + g'$ allows one to solve for g by writing $g' = Q - h_y$ and integrating.

The theorem above also has generalizations to higher dimensions. In three dimensions, a conservative vector field $\vec{F} = \langle P, Q, R \rangle$ with P, Q, R continuously differentiable satisfies $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. If $\vec{F}$ is continuously differentiable on a simply connected domain, then these equations will hold if and only if $\vec{F}$ is conservative. When we know that $\vec{F}$ is conservative, we can find a potential f by first integrating $f_x = P$ to get $f(x, y, z)$ up to a function g depending on y, z. Then, we can differentiate this with respect to y and integrate to get $g(y, z)$ up to a function $h(z)$. Finally, we do this one more time with respect to z to determine h up to a constant.

We use the symbol $\oint_C$ to signify that C is a closed curve (the integral is evaluated in the usual way). We know that if $\vec{F}$ is a conservative vector field then $\oint_C \vec{F} \cdot d\vec{r} = 0$. However, there is a more general statement.

Theorem (Green's)**.** If C is a positively oriented simple closed curve bounding a region R (this means that the boundary is traversed once counterclockwise) and P, Q are continuously differential in all of R, then

$$\oint_C P\ dx + Q\ dy = \iint_R (Q_x - P_y)\ dA.$$

This is only for closed curves! Also there is a generalization for the boundary of a domain that is not simply connected (so there are several boundary curves), but we need to orient the inner boundary curves in the opposite direction. One application is that for a region R bounded by a closed curve C, we have that

$$\text{area}(R) = \oint_C x\ dy = \oint_C -y\ dx = \oint_C \frac{x\ dy - y\ dx}{2}.$$

Sections 16.3 and 16.4 in Calculus, 7th Edition, by James Stewart