Summary for Lectures 8-9

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Applications of triple integrals

The applications of triple integrals are also similar to the applications of double integrals. For instance, the volume of a region is the integral of the function $f(x, y, z) = 1$ over the region and the mass is given by the integral of a mass density function $\rho(x, y, z)$. The average value of a function is

$$\bar{f} = \frac{1}{\text{vol}(R)} \iiint_R f(x, y, z)\ dV$$

and the center of mass $(\bar{x}, \bar{y}, \bar{z})$ is given by

$$\bar{x} = \frac{1}{\text{mass}(R)} \iiint_R x\rho\ dV$$

with similar formulas for the other variables.

Notice that the center of mass of a body with an axis of symmetry and constant density must lie on this axis. For example, the center of mass of a circular cylinder of constant density has its center of mass on the axis of the cylinder. In the same way, the center of mass of a spherically symmetric body of constant density is at the center of the sphere.

The formula for the moment of a solid R with mass density $\rho(x, y, z)$ about an axis in three dimensional space is

$$I = \iiint_R d^2(x, y, z)\rho(x, y, z)\ dV$$

where $d(x, y, z)$ is the distance to the axis.

Spherical coordinates

In addition to Cartesian and cylindrical coordinates, spherical coordinates are another useful coordinate system for computing triple integrals. They are related to Cartesian coordinates by

$$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$$

where ϕ (between 0 and π) is the angle down from the positive z-axis, ρ is the distance to the origin, and θ is the same angle that appears in cylindrical coordinates. Cylindrical coordinates and spherical coordinates are related by

$$r = \rho\sin\phi \qquad z = \rho\cos\phi \qquad \theta = \theta.$$

The Euclidean volume element in spherical coordinates is given by

$$dV = dx\ dy\ dz = \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta$$

and we almost always integrate in this order.

Sections 15.7, 15.8 and 15.9 in Calculus, 7th Edition, by James Stewart