**Summary for Lectures 15**

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The curl of a vector field is an operation that measures the failure of a vector field to be conservative. In two dimensions, the curl operation inputs a vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ and outputs a two variable function

$$\text{curl } \vec{F} = Q_x - P_y.$$

In three dimensions, the curl operation takes a vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + P\hat{k}$ and outputs another vector field

$$\text{curl } \vec{F} = \nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\hat{\imath} + (P_z - R_x)\hat{\jmath} + (Q_x - P_y)\hat{k}.$$

In both cases, we have a theorem that states that if $\vec{F}$ is continuously differentiable on a simply connected region then $\vec{F}$ is conservative if and only if curl $\vec{F} = 0$. Note that in the two dimensional case for a positively oriented simple closed curve $C$ bounding a region $R$, Green's theorem can be restated as

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \, dA.$$

We are also now considering the flux of a vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ in the plane across a curve $C$. This is given by

$$\int_C \vec{F} \cdot \hat{n} \, ds = \int_C -Q \, dx + P \, dy$$

where $\hat{n}$ is the unit normal vector to $C$. This measures the flow rate through $C$ if $\vec{F}$ is the velocity field of some liquid flow. Note that the equality $\hat{n} \, ds = \langle dy, -dx \rangle$ used above follows from the fact that $d\vec{r} = \hat{T} \, ds = \langle dx, dy \rangle$ and that $\hat{n}$ is perpendicular to $\hat{T}$.

We defined one more operation on vector fields. The divergence inputs a vector field $\vec{F}$ (in any number of coordinates) and outputs the sum of its partial derivatives. For instance, in dimension 2, we have

$$\text{div } \vec{F} = P_x + Q_y$$

if $\vec{F} = \langle P, Q \rangle$. Using this, Green's theorem can also be stated in the form

$$\oint_C \vec{F} \cdot \hat{n} \, ds = \iint_R \text{div } \vec{F} \, dA$$

where $\vec{F}$ is continuously differentiable and $C$ is a simple closed curve oriented counterclockwise which bounds the region $R$. Sometimes, the diverengence is also written as $\nabla \cdot \vec{F}$ to help us remember the formula. The geometric meaning of divergence is that it measures how $\vec{F}$ expands and contracts while curl measures how $\vec{F}$ spins.

Sections 16.5 in Calculus, 7th Edition, by James Stewart