Summary for Lectures 16-18

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Our final topic of this course will be to understand surface integrals. We want to integrate functions and vector fields over a surface S in space. Thus, we need to understand what an area form dS on a surface looks like (similar to ds in line integral). In particular, we want area$(S) = \iint_S dS$. To actually set this up and compute it, we need to parameterize the surface as

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

where (u,v) lie in some region R in the plane then we have

$$dS = |\vec{r}_u \times \vec{r}_v| \; du \; dv$$

and our integral will become a double integral over the region of space in which u and v vary. That is,

$$\iint_S f \; dS = \iint_R f(x(u,v), y(u,v), z(u,v)) \; |\vec{r}_u \times \vec{r}_v| \; du \; dv$$

where $(u,v) \in R$.

We can also use this to define the flux $\vec{F}$ of a vector field through a surface S. To set up this integral, we must choose a direction in which we will consider the flux positive (we had to do this also for curves in the plane). This amounts to choosing a direction that the unit normal vector $\hat{n}$ must point at each point. We call this choosing an orientation of S. The flux is then defined as

$$\iint_S \vec{F} \cdot \hat{n} \; dS$$

with dS as above. This integral again measures the flow rate through S if $\vec{F}$ is a velocity vector field. To compute, we again take a parameterization $\vec{r}(u,v)$. Then, we have that

$$\iint_S \vec{F} \cdot \hat{n} \; dS = \iint_S \vec{F}(u,v) \cdot \pm(\vec{r}_u \times \vec{r}_v) \; du \; dv$$

where the $\pm$ sign is determined by whether or not $\vec{r}_u \times \vec{r}_v$ points in the same direction as our choice of direction for the normal vector (you can check it at a single point). We sometimes use the shorthand notation $\hat{n} \; dS = d\vec{S}$ and write the integral as $\iint_S \vec{F} \cdot d\vec{S}$.

Sections 16.6and 16.7 in Calculus, 7th Edition, by James Stewart