Summary for Lectures 19-21

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We are continuing to study the flux of a vector field $\vec{F}$ through a surface S which is given by

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\ dS = \iint_R \vec{F} \cdot \pm(\vec{r}_u \times \vec{r}_v)\ du\ dv$$

with $\vec{r}(u,v)$ a parameterization of S with u, v varying in the region R and where the sign depends on the choice of the direction of the unit normal $\hat{n}$ (orientation). It is important to note that $\hat{n} \neq \pm(\vec{r}_u \times \vec{r}_v)$ and $dS \neq du\ dv$, but rather $\hat{n} = \pm(\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$ and $dS = |\vec{r}_u \times \vec{r}_v|\ du\ dv$.

Similar to Green's theorem for flux in the plane, we have the following theorem on flux integrals in space.

Theorem (Divergence Theorem)**.** If S is a closed surface bounding a region D in space and oriented with $\hat{n}$ pointing outwards of D, and $\vec{F}$ is a continuously differentiable vector field on D, then

$$\oiint_S \vec{F} \cdot \hat{n}\ dS = \iiint_D \text{div}\ \vec{F}\ dV.$$

The final topic of this course is Stokes' theorem. Stokes' theorem applies to a surface S in space with boundary curve C. In order to apply the theorem, S and C must have compatible orientation according to the right hand rule. If you point your thumb in the direction of C and the rest of your fingers in to S, then your palm will be pointing in the same direction as $\hat{n}$. The statement of the theorem is as follows.

Theorem (Stokes)**.** Suppose F is a continuously differentiable vector field near S. If C is a closed curve bounding a surface S in space with compatible orientations, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot \hat{n}\ dS.$$

Sections 16.8 and 16.9 in Calculus, 7th Edition, by James Stewart