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Lecture 4-5 Linear combinations, spans and linear Independence

New perspective on linear systems : vector equations

Augmented matrix m rows $\left[\begin{array}{cccc|c} | & | & & | & \\ \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_n & \bar{b} \\ | & | & & | & \end{array}\right]$ where $\bar{v}_i, \bar{b} \in \mathbb{R}^m$

n+1 columns

linear system is same as vector equation

$$x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2 + \dots + x_n \bar{v}_n = \bar{b}$$

scalar variables (x_1, x_2, x_n)

<u>Example</u> Solve vector equation

$$x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

<u>Solution:</u> <u>Notice</u> $x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}$

Two vectors are equal if and only if all corresponding components are equal.

given equation is equivalent to linear system

$$\begin{cases} x_1 = 2 \\ x_1 + x_2 = -2 \end{cases}$$

which is equivalent to

augmented matrix $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & -2 \end{array}\right]$

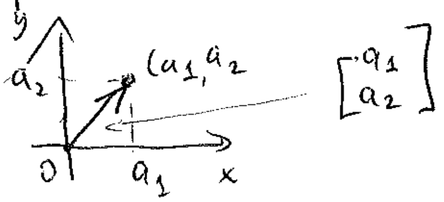
1. Bring augmented matrix into RREF

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -4 \end{array}\right] \to \begin{array}{l} x_1 = 2 \\ x_2 = -4 \end{array}$$

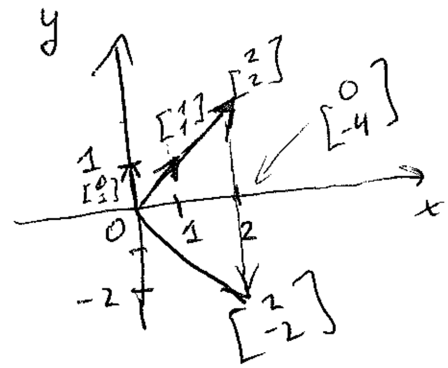
replace second row by second row minus first row

② Geometric picture

There exists a correspondence between vectors $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and points (a_1, a_2) in the plane $\mathbb{R}^2$.



For our example



<u>Def.</u> A vector $\bar{u}$ is a <u>linear combination</u> of vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ if there are numbers $x_1, x_2, \ldots, x_k$ such that $\bar{u} = x_1 \bar{v}_1 + x_2 \bar{v}_2 + \ldots + x_k \bar{v}_k$

From previous exercise, we can say $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as

$$\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \underset{x_1}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underset{x_2}{(-4)} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

<u>Notice!</u> $\bar{u}$ is a linear combination of $\bar{v}_1, \ldots, \bar{v}_k$ if and only if the linear system

$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_k & \bar{u} \\ | & | & & | & | \end{array}\right]$$ is consistent.

(3) *Example*
For which values of c is $\bar{u}$ a linear combination of $\bar{v}_1, \bar{v}_2$?

$$\bar{u} = \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix}, \quad \bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \bar{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Solution: 1. Is the following augmented matrix consistent?

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & c \end{array}\right]$$

2. Bring to REF (row echelon form)

$$\xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & c \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & c+2 \end{array}\right]$$

$c+2 = 0 \iff \boxed{c = -2} \to$ consistent

$c+2 \neq 0 \iff \boxed{c \neq -2} \to$ pivot in the last column of REF $\to$ no solutions, i.e., inconsistent.

Def. Span of m-vectors $\bar{v}_1, \dots, \bar{v}_k$ is

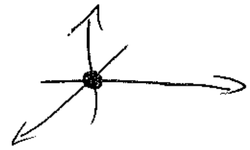
$\text{Span}\{\bar{v}_1, \dots, \bar{v}_k\} = \{$all linear combinations of $\bar{v}_1, \dots, \bar{v}_k\} = \{a_1\bar{v}_1 + \dots + a_k\bar{v}_k$, where $a_1, \dots, a_k$ - any numbers$\}$

Notice! $\text{Span}\{\bar{v}_1, \dots, \bar{v}_k\} \underset{\text{subset}}{\subseteq} \mathbb{R}^m$ $(= \{$all m-vectors$\})$

Notice! $\bar{u}$ is in $\text{Span}\{\bar{v}_1, \dots, \bar{v}_k\}$ if and only if $\bar{u}$ is a linear combination of $\bar{v}_1, \dots, \bar{v}_k$

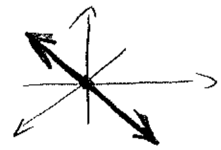
Examples of a span in $\mathbb{R}^m$

1) $\{\bar{0}\}$ origin alone



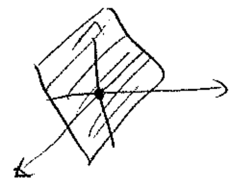
$a \cdot \bar{0} = \bar{0}$ for any number a

2) line through origin



$\{c \cdot \bar{u}$, where c – number$\}$

3) plane through origin



$\{a \cdot \bar{v}_1 + b\,\bar{v}_2$, where a, b – numbers$\}$

<u>Def.</u> We say that vectors $\bar{v}_1, \dots, \bar{v}_k$ span $\mathbb{R}^m$ if $\text{Span}\{\bar{v}_1, \dots, \bar{v}_k\} = \mathbb{R}^m$, i.e., every vector in $\mathbb{R}^m$ is a linear combination of $\bar{v}_1, \dots, \bar{v}_k$.

<u>Notice!</u> m-vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ span $\mathbb{R}^m$

$$\Updownarrow$$

$$\left[\begin{array}{cccc|c} | & | & & | & | \\ \bar{v}_1 & \bar{v}_2 & \dots & \bar{v}_k & \bar{u} \\ | & | & & | & | \end{array}\right] \text{ is consistent for any } \bar{u} \in \mathbb{R}^m$$

$$\Updownarrow$$

coefficient matrix $\begin{bmatrix} | & | & & | \\ \bar{v}_1 & \bar{v}_2 & \dots & \bar{v}_k \\ | & | & & | \end{bmatrix}$ has a pivot in each row, i.e., we have m pivots.

<u>Example</u> Let $\bar{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\bar{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$.

For what values of h is $\bar{y}$ in the plane generated by $\bar{v}_1$ and $\bar{v}_2$?

<u>Solution!</u> $\bar{y}$ is in the plane generated by $\bar{v}_1$ and $\bar{v}_2$ if and only if there exists x_1 and x_2 such that

$$\bar{y} = x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2$$

(5)
$$\left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{array}\right] \xrightarrow{R_3 \to R_3+2R_1} \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & 2h-3 \end{array}\right] \to$$

$$\xrightarrow{R_3 \to R_3-2R_2} \left[\begin{array}{cc|c} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h+7 \end{array}\right]$$

There exist solution if and only if $2h+7=0$

$\Longleftrightarrow$ $\boxed{h = -7/2}$

Def. $\bar{v}_1, \dots, \bar{v}_k$ are linearly dependent if there are numbers $x_1, \dots, x_k$ not all 0 (at least one x_i is not zero) such that $x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2 + \dots + x_k \bar{v}_k = \bar{0}$

Otherwise, $\bar{v}_1, \dots, \bar{v}_k$ are called linearly independent.

Notice! $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ are linearly independent means whenever $x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2 + \dots + x_k \cdot \bar{v}_k = \bar{0}$ we have $x_1 = x_2 = \dots = x_k = 0$.

Consider a system

$$\left[\begin{array}{cccc|c} | & | & & | & 0 \\ \bar{v}_1 & \bar{v}_2 & \dots & \bar{v}_k & \vdots \\ | & | & & | & 0 \end{array}\right]$$

(it is called homogeneous system)

$(0, \dots, 0)$ is a solution for any $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$.

Therefore, the system can have either the unique solution (i.e., only $(0, 0, \dots, 0)$) or infinitely many (i.e., in particular, nonzero solution $(a_1, a_2, \dots, a_k)$ with at least one $a_i \neq 0$).

⑥ Notice!

1) $\overline{v_1}, \dots, \overline{v_k}$ are linearly independent

$\Updownarrow$

$$\left[\begin{array}{cccc|c} | & | & & | & 0 \\ \overline{v_1} & \overline{v_2} & \dots & \overline{v_k} & 0 \\ & & & & \vdots \\ | & | & & | & 0 \end{array}\right]$$ has <u>unique</u> solution $(0, 0, \dots, 0)$

2) $\overline{v_1}, \dots, \overline{v_k}$ are linearly dependent

$\Updownarrow$

$$\left[\begin{array}{cccc|c} | & | & & | & 0 \\ \overline{v_1} & \overline{v_2} & \dots & \overline{v_k} & 0 \\ & & & & \vdots \\ | & | & & | & 0 \end{array}\right]$$ has <u>infinitely many</u> solutions

<u>Example</u> Consider list of vectors in $\mathbb{R}^5$

$$\overline{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \overline{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \overline{v_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \overline{v_4} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \overline{v_5} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \overline{v_6} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Find the <u>smallest</u> k such that $\overline{v_1}, \overline{v_2}, \dots, \overline{v_k}$ is linearly dependent.

<u>Solution</u>: Place vectors in columns of coefficient matrix.

Consider augmented matrix

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \overline{v_1} & \overline{v_2} & \overline{v_3} & \overline{v_4} & \overline{v_5} & \overline{v_6} & \end{array}\right] \longrightarrow$$

Seek solution $(x_1, \dots, x_6)$ where $x_k \neq 0$ but $x_{k+1} = \dots = x_6 = 0$.

Want smallest such solution.

Idea: put in REF and k is the index of the first column corresponding to free variable (no pivot)

(7) ← pivot

$$\xrightarrow[R_5 \to R_5 - R_1]{R_3 \to R_3 - R_1} \left[\begin{array}{cccccc|c} \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccccc|c} \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_4 \to R_4 - R_3} \left[\begin{array}{cccccc|c} \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{2} & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

↑ first free column

x_5, x_6 are free variables. We can set $x_5 = 1$ and $x_6 = 0$

We can solve for $\underbrace{x_1, x_2, x_3, x_4}_{\text{basic variables}}$ → $\left(-1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 1, 0\right)$ is a solution

In particular, it means $-\bar{v}_1 + \frac{1}{2}\bar{v}_2 + \frac{1}{2}\bar{v}_3 - \frac{1}{2}\bar{v}_4 + \bar{v}_5 = 0$

Therefore, $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4, \bar{v}_5$ are linearly dependent.

<u>Notice!</u> $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ are linearly independent as we have pivots in 1, 2, 3, 4 columns.

<u>Notice!</u> <u>Thm</u> Any set $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

(We have less equations than variables in corresponding linear system ⇒ we have free variables ⇒ ⇒ nonzero solution)

(8) Matrices

Let A - $m \times n$ matrix
$\bar{x}$ - n-vector

Multoplication of a matrix and vector

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} \cdot x_1 + a_{12} x_2 + \dots + a_{1n} x_n \\ a_{21} \cdot x_1 + a_{22} x_2 + \dots + a_{2n} x_n \\ \vdots \\ a_{m1} \cdot x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \end{bmatrix}$$

denote columns $\bar{a}_1$, $\bar{a}_2$, ..., $\bar{a}_n$

$$= x_1 \cdot \bar{a}_1 + x_2 \cdot \bar{a}_2 + \dots + x_n \cdot \bar{a}_n$$

New perspective on linear systems: matrix equation

$$A\bar{x} = \bar{b}$$

$A\bar{x} = \bar{0}$ — <u>homogeneous</u> matrix equation

<u>Example</u> Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.
Show that the equation $A\bar{x} = \bar{b}$ doesn't have a solution for all possible $\bar{b}$, and find $\bar{b}$ for which $A\bar{x} = \bar{b}$ does have a solution.

<u>Solution:</u> Augmented matrix

$$\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{array}\right] \xrightarrow{R_2 \to R_2 + 3R_1} \left[\begin{array}{cc|c} 2 & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{array}\right]$$

→ Bring into REF

If $b_2 + 3b_1 = 0$, then system is consistent.
But, if $b_2 = 1$ and $b_1 = 0$, then $b_2 + 3b_1 \neq 0 \to$ for $\bar{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ no solution.
Any vector $\begin{bmatrix} b_1 \\ -3b_1 \end{bmatrix}$, where b_1 - any number, makes system consistent.