1

Matrices

Lecture 6-7

Let A – $m \times n$ matrix

$\bar{x}$ – n-vector

Multoplication of a matrix and vector

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}\cdot x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}\cdot x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}\cdot x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

denote $\bar{a}_1 \quad \bar{a}_2 \quad \dots \quad \bar{a}_n$

$$= x_1\cdot \bar{a}_1 + x_2 \cdot \bar{a}_2 + \dots + x_n \cdot \bar{a}_n$$

New perspective on linear systems: matrix equation

$$A\bar{x} = \bar{b}$$

$A\bar{x} = \bar{0}$ — <u>homogeneous</u> matrix equation

<u>Example</u> Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Show that the equation $A\bar{x} = \bar{b}$ doesn't have a solution for all possible $\bar{b}$, and find $\bar{b}$ for which $A\bar{x} = \bar{b}$ does have a solution.

<u>Solution:</u> Augmented matrix

$$\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{array}\right] \xrightarrow{R_2 \to R_2 + 3R_1} \left[\begin{array}{cc|c} 2 & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{array}\right]$$

→ Bring into REF

If $b_2 + 3b_1 = 0$, then system is consistent.

But, if $b_2 = 1$ and $b_1 = 0$, then $b_2 + 3b_1 \neq 0 \Rightarrow$ for $\bar{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ no solution.

Any vector $\begin{bmatrix} b_1 \\ -3b_1 \end{bmatrix}$, where b_1 – any number, makes system consistent.

② Solution sets of linear systems.

a) Assume in the previous example $\bar{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e. system is consistent and

REF $\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{R_1}{2}}$ RREF $\left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

x_2 - free variable, x_1 - basic variable

$x_1 - 1/2\, x_2 = 0 \;\to\; x_1 = \frac{1}{2} x_2$

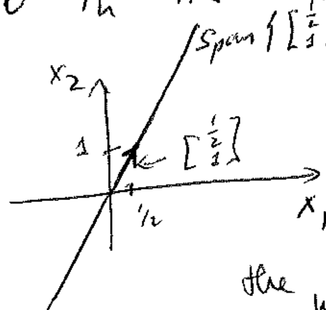
Solution $\to$ $\bar{X} = \begin{bmatrix} \frac{1}{2} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \cdot x_2$, where $x_2 \in \mathbb{R}$ or $\bar{x} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \cdot t$, where $t \in \mathbb{R}$ $\leftarrow$ parametric vector form

Here x_2 is factored out of the expression for the general solution vector.

Every solution is a multiple of vector $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

Trivial solution if $x_2 = 0$.

Geometrically, the solution set is a line through $\bar{0}$ in $\mathbb{R}^2$



In general, the solution for homogeneous system with infinitely many solutions can write the solution as sum of terms that are constant vectors times a free variable.

③ b) Assume that $\bar{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

REF $\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{R_1}{2}}$ RREF $\left[\begin{array}{cc|c} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$

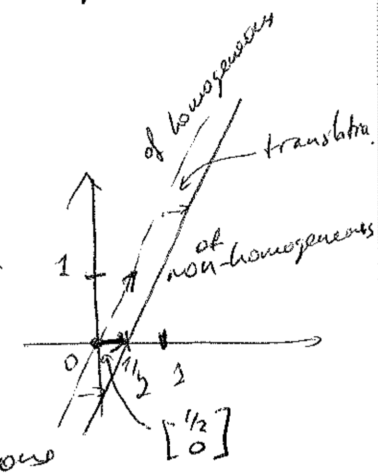
x_2 - free, x_1 - basic

$x_1 - 1/2 x_2 = 1/2 \longrightarrow x_1 = 1/2 x_2 + 1/2$

solution: $\bar{x} = \begin{bmatrix} \frac{1}{2}x_2 + \frac{1}{2} \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \cdot x_2 + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

separate free variables and constant vector

$$\bar{x} = \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \cdot t}_{\substack{\text{solution of homogeneous} \\ \text{with the same matrix } A \\ (A\bar{x} = \bar{0})}} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}}_{\substack{\text{particular solution of} \\ \text{non-homogeneous with the matrix } A \\ (A\bar{x} = \bar{b})}}, \text{ where } t \in \mathbb{R}$$



is equivalent to a system

$$\begin{cases} 2x_1 - x_2 = 1 \\ 0 \cdot x_1 + 0 \cdot x_2 = 0 \end{cases} \longrightarrow 2x_1 - x_2 = 1$$

if $x_1 = 1/2$ and $x_2 = 0$,

then $2 \cdot \frac{1}{2} - 0 = 1 - 0 = 1$ ✓

satisfy the equation

<u>Thm</u> Suppose the equation $A\bar{x} = \bar{b}$ is consistent for some given $\bar{b}$ and $\bar{p}$ is a solution. Then the solution set of $A\bar{x} = \bar{b}$ is the set of all vectors of the form $\bar{x} = \bar{p} + \bar{v}_h$, where $\bar{v}_h$ is any solution of the homogeneous equation $A\bar{x} = \bar{0}$.

(4) Example

Describe all solutions of $A\bar{x} = \bar{0}$ in parametric vector form, where A is row equivalent to the given matrix

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

Solution: Augmented matrix of $A\bar{x} = \bar{0}$ is

(adding zeros, multiplying zero by a number = zero)

in REF $\rightarrow \left[\begin{array}{cccc|c} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$

$$\xrightarrow[R_1 \to R_1 - 3R_2]{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 9 & -8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

$$\rightarrow \begin{cases} x_1 + 9x_3 - 8x_4 = 0 \\ x_2 - 4x_3 + 5x_4 = 0 \\ x_3, x_4 - \text{free} \end{cases}$$

$$\rightarrow \bar{x} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \cdot x_3 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \cdot x_4 \\ x_4 \end{bmatrix}$$

Solution: $\bar{x} = \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} \cdot x_3 + \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix} \cdot x_4$, where $x_3, x_4 \in \mathbb{R}$ free variables.

Geometrically, is a plane in $\mathbb{R}^4$

(5) <u>Linear transformations</u>

"It's not about perfect. It's about effort. And when you bring that effort every single day, that's where transformation happens. That's how change occurs."
— Jillian Michaels

Think of the coefficient matrix
$A = \begin{bmatrix} | & | & & | \\ \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_n \\ | & | & & | \end{bmatrix}$ as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$
where $\bar{a}_i$ - m-vectors

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

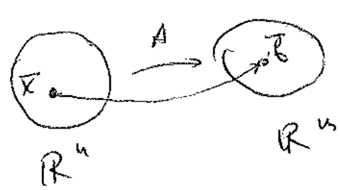
input, <u>domain</u> — output, target

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto A\bar{x} = x_1\bar{a}_1 + x_2\bar{a}_2 + \dots + x_n\bar{a}_n = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

Interpretation of solving $A\bar{x} = \bar{b}$

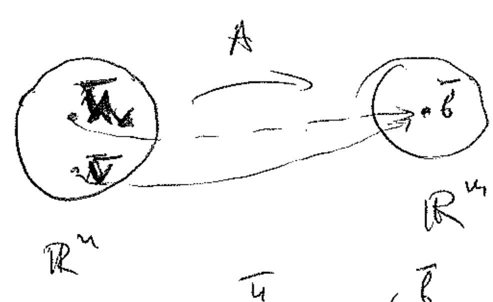
<u>Existence?</u>

Is there $\bar{x} \in \mathbb{R}^n$ such that A takes $\bar{x}$ to $\bar{b} \in \mathbb{R}^m$?



<u>Uniqueness?</u>

How many $\bar{x} \in \mathbb{R}^n$ does A take to $\bar{b} \in \mathbb{R}^m$?



Example

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\bar{u}} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\bar{b}}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\bar{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(6) **Def** Given $A = \begin{bmatrix} | & | & & | \\ \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_n \\ | & | & & | \end{bmatrix}$, where $\bar{a}_i$ - m-vectors,

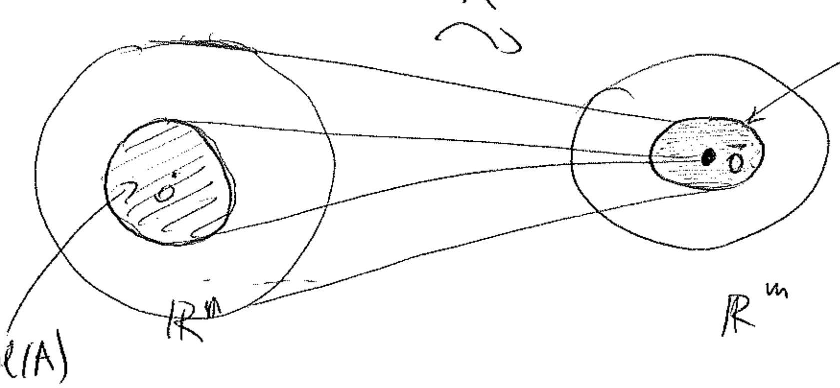
Think of it as a map $\mathbb{R}^n \to \mathbb{R}^m$

1) Image/Range of A

$\text{Image}(A) = \text{Range}(A) = \{\text{all } \bar{y} \in \mathbb{R}^m$ s.t. there is $\bar{x} \in \mathbb{R}^n$ with $A\bar{x} = \bar{y}\}$

2) Kernel/Null space of A

$\text{Kernel}(A) = \text{NullSpace}(A) = \{\text{all } \bar{x} \in \mathbb{R}^n$ s.t. $A\bar{x} = \bar{0}\}$



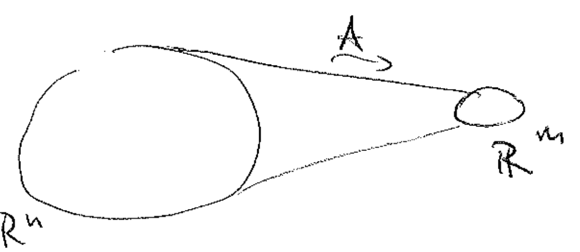
Recall! $A\bar{0} = \bar{0}$

Notice 1) $\text{Imag}(A) = \{\text{all } \bar{b} \in \mathbb{R}^m$ s.t. $[A \mid \bar{b}]$ has solution$\}$

2) $\text{Kernel}(A) = \{\text{all } \bar{x} \in \mathbb{R}^n$ that solve $[A \mid \bar{0}]\}$

Def $A: \mathbb{R}^n \to \mathbb{R}^m$, A - matrix

1) A is onto/surjective if $\text{Image}(A) = \mathbb{R}^m$



(7) 2) A is one-to-one / injective if whenever $A\bar{u} = A\bar{v}$ we have $\bar{u} = \bar{v}$

<u>Lemma</u> A is one-to-one if and only if $\text{Kernel}(A) = \{\bar{0}\}$.

1) If A is one-to-one, $\bar{x} \in \text{Kernel}(A)$, i.e., $A\bar{x} = \bar{0}$.
we know that $A\bar{0} = \bar{0}$. Therefore, $\bar{x} = \bar{0}$ by definition of one-to-one.

2) Suppose $\bar{u}, \bar{v} \in \mathbb{R}^n$ are such that
$$A\bar{u} = A\bar{v}.$$
Consider vector $\bar{u} - \bar{v}$. $A(\bar{u} - \bar{v}) = A\bar{u} - A\bar{v} = \bar{0}$.
Therefore, $\bar{u} - \bar{v} \in \text{Kernel}(A)$
If $\text{Kernel}(A) = \{\bar{0}\}$, then $\bar{u} - \bar{v} = \bar{0} \Rightarrow \bar{u} = \bar{v} \Rightarrow$
$\Rightarrow$ A is one-to-one.

<u>Summary</u>

1) A onto $\Longleftrightarrow$ for all $\bar{b} \in \mathbb{R}^m$ we can solve $[A \mid \bar{b}]$
$\Longleftrightarrow$ columns of A span $\mathbb{R}^m$
$\Longleftrightarrow$ pivot in each row of A

2) A one-to-one $\Longleftrightarrow$ $\bar{0}$ is only solution of $[A \mid \bar{0}]$
$\Longleftrightarrow$ columns of A linearly independent
$\Longleftrightarrow$ pivot in each column of A.

8 **Example**

For what values of c is $A = \begin{bmatrix} 1 & 0 & 1 \\ c & 1 & 0 \\ 0 & c & c \end{bmatrix}$ onto, one-to-one? $(A: \mathbb{R}^3 \to \mathbb{R}^3)$

Solution: Bring to REF

$$\begin{bmatrix} \boxed{1} & 0 & 1 \\ c & 1 & 0 \\ 0 & c & c \end{bmatrix} \xrightarrow{R_2 \to R_2 - cR_1} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -c \\ 0 & c & c \end{bmatrix} \xrightarrow{R_3 \to R_3 - cR_2} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -c \\ 0 & 0 & c+c^2 \end{bmatrix}$$

1) $c + c^2 = 0 \iff c(c+1) = 0$ → $c = 0$ or $c = -1$ → no pivot in third row or column → neither onto, nor one-to-one

2) $c + c^2 \neq 0$, i.e., $c \neq 0$ & $c \neq -1$ → pivot in each column and each row of A → both onto and one-to-one.

~~No more such~~

b) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & c \end{bmatrix}$. Same question. $(A: \mathbb{R}^3 \to \mathbb{R}^2)$

→ REF $\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & c \end{bmatrix}$

pivot in each row of A → onto

no pivot in third column (for any c) → not one-to-one

~~c) Let A = [1 ...] Same question (A ...)~~

c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ $(A: \mathbb{R}^2 \to \mathbb{R}^3)$

pivot in each column → one-to-one

no pivot in third row → not onto

What is special about map $\mathbb{R}^n \to \mathbb{R}^m$ given by a matrix A?

Properties of matrix-vector product!

$$A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$$
$$A(c\bar{u}) = c(A\bar{u})$$
, where $\bar{u}, \bar{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$ – scalar.

<u>Def</u> A <u>linear transformation</u> $T: \mathbb{R}^n \to \mathbb{R}^m$ is any map that satisfies both properties.

1) $T(\bar{u}+\bar{v}) = T\bar{u} + T\bar{v}$
2) $T(c\bar{u}) = c\,T(\bar{u})$

, where $\bar{u}, \bar{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$

<u>Notice!</u> What is $T(\bar{0})$?

$$\underline{T(\bar{0})} = T(\bar{0}+\bar{0}) = \underline{T(\bar{0}) + T(\bar{0})}$$
$$T(\bar{0}) = 2T(\bar{0}) \implies T(\bar{0}) = \bar{0}$$

<u>Important!</u>

Any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is given by some $m \times n$ matrix A!

Let $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ... $\bar{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

Any vector in $\mathbb{R}^n$ $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\bar{e}_1 + v_2\bar{e}_2 + \dots + v_n\bar{e}_n$

If we know $T(\bar{e}_i)$, then we know $T(\bar{v}) = v_1 T(\bar{e}_1) + v_2 T(\bar{e}_2) + \dots + v_n T(\bar{e}_n)$

Matrix A of T is: $A = \begin{bmatrix} | & | & & | \\ T(\bar{e}_1) & T(\bar{e}_2) & \dots & T(\bar{e}_n) \\ | & | & & | \end{bmatrix}$