Lectures 1-2. Systems of linear equations.

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We use the following notations. $\mathbb{R}$ is the set of real numbers. The expression "$t \in \mathbb{R}$" means that t is a real number.

Let n, m be natural numbers.

Definition 1. A linear equation in n variables is any equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n, b$ are constants and $x_1, x_2, \ldots, x_n$ are variables.

Definition 2. A linear system of m equations in n variables is a group of m linear equations in n variables, i.e. it is a system of equations that can be written in the form

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where the x_i are variables and the a_{ij} and b_i are constants.

Definition 3. The solution set of a linear system in n variables is a set of ALL n-tuples $(s_1, s_2, \ldots, s_n)$ of numbers that solve the system, i.e. each equation in the system becomes a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

Definition 4. Two linear systems are equivalent if they have the same solution set.

Theorem 5. *A system of linear equations has either*

1. *no solutions (inconsistent linear system) or*

2. *unique solution (consistent linear system) or*

3. *many solutions, in fact in this case there will always be infinitely many solutions (consistent linear system).*

Linear systems can be solved directly by substitution, but when many variables are involved, it is often quicker and more convenient to represent the system as an augmented matrix as follows.

$$\left(\begin{array}{ccc|c} a_{11} & \ldots & a_{1n} & b_1 \\ a_{21} & \ldots & a_{2n} & b_2 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \ldots & a_{mn} & b_m \end{array}\right)$$

The system can be solved by performing elementary row operations that preserve the solution set.

Elementary row operations:

1. Add a multiple of a row to another;
2. Interchange two rows;
3. Scale row by nonzero number.

Row echelon form (REF) of a matrix:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry/pivot of a row is in a column to the right of the leading entry of the row above.
3. All entries in a column below a leading entry are zeros.

Reduced row echelon form (RREF) of a matrix:

1. The matrix is in REF.
2. The leading entry in each nonzero row is 1.
3. Each leading 1 is the only nonzero entry in its column.

There is no more than one pivot in any row or any column. A pivot column (i.e., a column that contains a pivot) of an augmented matrix of a linear system corresponds to a basic variable. Other columns correspond to free variables.

Theorem 6. *1. A linear system is inconsistent if and only if there is a pivot in the last column of the augmented matrix.*

2. A consistent linear system without free variables has a unique solution.

3. A consistent linear system with free variables has infinitely many solutions.

To answer the question if a linear system is consistent, bring the augmented matrix of the system into REF and make sure that there is no rows of form

$$\left(\begin{array}{cccc|c} 0 & 0 & \dots & 0 & b \end{array}\right)$$

where b is a NONZERO constant.

Theorem 7. *Given any matrix, it is possible to put it in REF by elementary row operations. Moreover, it is possible to put it in RREF uniquely.*

Steps to put a matrix in RREF:

1. Find left most nonzero column. The pivot position is the first spot in this column.

2. Find nonzero entry of this column. You want it to be in pivot position, if it is not, then interchange rows.
3. Create zeros in all positions below the pivot by adding an appropriate multiple of the first row to the rows below.
4. Repeat steps 1-3 for the submatrix (i.e., a matrix that you obtain by removing the first row and all columns to the left of the first pivot column, including the first pivot column).
5. Scale rows with pivots to make pivots equal to 1.
6. Add appropriate multiples of the rows with pivots to all other starting from the rightmost pivot.

To put a matrix in REF, just do steps 1-4.