<u>Lectures 3.</u> Linear Combinations and Spans.

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Denote by $\mathbb{R}^m$ the set of m-vectors with entries being real numbers.

Let $\overline{u} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$ and $\overline{v} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ be m-vectors, where $a_1, \dots, a_m, b_1, \dots, b_m$ - numbers.

<u>Basic Operations</u>

1. $\overline{u} + \overline{v} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{pmatrix}$

2. Let c be a number. $c\overline{u} = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_m \end{pmatrix}$

3. Dot product: $\overline{u} \cdot \overline{v} = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m$

There are three ways to present a linear system: equations, augmented matrices and vector equations.

Definition 1. A vector $\overline{u}$ is a <u>linear combination</u> of vectors $\overline{v_1}, \overline{v_2}, \dots, \overline{v_k}$ if there are numbers $x_1, x_2, \dots, x_k$ such that $\overline{u} = x_1\overline{v_1} + x_2\overline{v_2} + \cdots + x_k\overline{v_k}$.

Definition 2. A <u>*Span* of m-vectors</u> $\overline{v_1}, \overline{v_2}, \dots, \overline{v_k}$ is a collection of all linear combinations of $\overline{v_1}, \overline{v_2}, \dots, \overline{v_k}$.

Lemma 3. *A vector* $\overline{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$ *is in Span*$\{\overline{v_1} = \begin{pmatrix} v_1^1 \\ v_1^2 \\ \vdots \\ v_1^m \end{pmatrix}, \overline{v_2} = \begin{pmatrix} v_2^1 \\ v_2^2 \\ \vdots \\ v_2^m \end{pmatrix}, \dots, \overline{v_k} = \begin{pmatrix} v_k^1 \\ v_k^2 \\ \vdots \\ v_k^m \end{pmatrix}\}$ *if and only if*

$$\left(\begin{array}{cccc|c} v_1^1 & v_2^1 & \dots & v_k^1 & u_1 \\ v_1^2 & v_2^2 & \dots & v_k^2 & u_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_1^m & v_2^m & \dots & v_k^m & u_m \end{array}\right) \quad \textit{is consistent.}$$

Definition 4. Vectors $\overline{v_1}, \overline{v_2}, \dots, \overline{v_k}$ span $\mathbb{R}^m$ if *Span*$\{\overline{v_1}, \overline{v_2}, \dots, \overline{v_k}\} = \mathbb{R}^m$, i.e., every vector in $\mathbb{R}^m$ is a linear combination of $\overline{v_1}, \overline{v_2}, \dots, \overline{v_k}$.