Homework 3 - Solutions

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“(*)” means that the problem is optional.

1. (Section 1.4 Exercise 9) Write the system first as a vector equation and then as a matrix equation.

\[
\begin{align*}
3x_1 + x_2 - 5x_3 &= 9 \\
x_2 + 4x_3 &= 0
\end{align*}
\]

Solution. Vector equation:

\[
\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} -5 \\ 4 \end{pmatrix} x_3 = \begin{pmatrix} 9 \\ 0 \end{pmatrix}
\]

Matrix equation:

\[
\begin{pmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix}
\]

\[
\square
\]

2. Suppose function \( T(\bar{x}) \) is a linear transformation from \( \mathbb{R}^2 \to \mathbb{R}^3 \) with the standard matrix \( A = \begin{bmatrix} 4 & 0 \\ 5 & 8 \\ 6 & 6 \end{bmatrix} \). Is \( T(\bar{x}) \) one-to-one? Is \( T(\bar{x}) \) onto?

Solution. Bring \( A \) into row echelon form.

\[
\begin{pmatrix} 4 & 0 \\ 5 & 8 \\ 6 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 5 & 8 \\ 6 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 8 \\ 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

We have a pivot in each column, therefore \( T \) is one-to-one. We have no pivot in the third row, therefore \( T \) is not onto.

\[
\square
\]

3. Given that

\[
S = \left\{ \begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \bar{v} \right\}
\]

which of the following choices will make \( S \) linearly independent? Explain your answer.

\[
(a) \quad \bar{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad (b) \quad \bar{v} = \begin{bmatrix} 3 \\ -7 \\ 11 \end{bmatrix}; \quad (c) \quad \bar{v} = \begin{bmatrix} 20 \\ -50 \\ 80 \end{bmatrix}; \quad (d) \quad \bar{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.
\]
Solution. Let \( \bar{a} = \begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix} \) and \( \bar{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \).

(a) gives linear dependent as \( \bar{v} = 0 \cdot \bar{a} + 0 \cdot \bar{b} \)

(b) gives linear dependent as \( \bar{v} = 1 \cdot \bar{a} + 1 \cdot \bar{b} \)

(c) gives linear dependent as \( \bar{v} = 10 \cdot \bar{a} + 0 \cdot \bar{b} \)

We show that (d) makes \( S \) linearly independent. Bring the matrix with columns \( \bar{a}, \bar{b}, \bar{v} \) into row echelon form.

\[
\begin{pmatrix}
2 & 1 & 2 \\
-5 & -2 & -1 \\
8 & 3 & 3
\end{pmatrix} \sim
\begin{pmatrix}
2 & 1 & 2 \\
0 & 1/2 & 4 \\
0 & -1 & -5
\end{pmatrix} \sim
\begin{pmatrix}
2 & 1 & 2 \\
0 & 1 & 2 \\
0 & 0 & 3
\end{pmatrix}
\]

We have pivot in each column, therefore vectors are linearly independent.

4. (Section 1.4 Exercise 16) Let \( A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix} \) and \( \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \). Show that the equation \( A\bar{x} = \bar{b} \) does not have a solution for all possible \( \bar{b} \), and describe the set of all \( \bar{b} \) for which \( A\bar{x} = \bar{b} \) does have a solution.

Solution. Bring the corresponding augmented into row echelon form.

\[
\begin{pmatrix}
1 & -3 & -4 & b_1 \\
-3 & 2 & 6 & b_2 \\
5 & -1 & -8 & b_3
\end{pmatrix} \sim
\begin{pmatrix}
1 & -3 & -4 & b_1 \\
0 & -7 & -6 & b_2 + 3b_1 \\
0 & 14 & 12 & b_3 - 5b_1
\end{pmatrix} \sim
\begin{pmatrix}
1 & -3 & -4 & b_1 \\
0 & 0 & 0 & b_2 + 3b_1 \\
0 & 0 & 0 & b_1 + 2b_2 + b_3
\end{pmatrix}
\]

If \( b_1 + 2b_2 + b_3 \neq 0 \), then the system is inconsistent, i.e., there are no solutions. For example, there are no solutions for \( \bar{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).

If \( b_1 + 2b_2 + b_3 = 0 \), then the system is consistent. For any \( \bar{b} = \begin{pmatrix} -2b_2 - b_3 \\ b_2 \\ b_3 \end{pmatrix} \), where \( b_2, b_3 \in \mathbb{R} \), the system is consistent.

5. (Section 1.5 Exercise 16) Describe the solutions of the following system in parametric vector form, and give a geometric description of the solution set and compare it to the solutions of the corresponding homogeneous system.

\[
x_1 + 3x_2 - 5x_3 = 4 \\
x_1 + 4x_2 - 8x_3 = 7 \\
-3x_1 - 7x_2 + 9x_3 = -6
\]
Solution. Bring the corresponding row echelon form into reduced row echelon form.

\[
\begin{pmatrix}
1 & 3 & -5 & 4 \\
1 & 4 & -8 & 7 \\
-3 & -7 & 9 & -6
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 3 & -5 & 4 \\
0 & 1 & -3 & 3 \\
0 & 2 & -6 & 6
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 3 & -5 & 4 \\
0 & 1 & -3 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Notice that \(x_1, x_2\) are the basic variables and \(x_3\) is the free variable. The solution is \(\bar{x} = \begin{pmatrix}-5 - 4x_3 \\ 3 + 3x_3 \\ x_3\end{pmatrix}\). The solution in the parametric form is \(\bar{x} = \begin{pmatrix}-5 \\ 3 \\ 0\end{pmatrix} + \begin{pmatrix}-4 \\ 3 \\ 1\end{pmatrix}x_3\). The solution is a line through the point \(\begin{pmatrix}-5 \\ 3 \\ 0\end{pmatrix}\) in \(\mathbb{R}^3\). It is a translation of the solution of the corresponding homogeneous solution by the vector \(\begin{pmatrix}-5 \\ 3 \\ 0\end{pmatrix}\).

6. (Section 1.5 Exercise 37) Construct a \(2 \times 2\) matrix \(A\) such that the solution set of the equation \(Ax = \vec{0}\) is the line in \(\mathbb{R}^2\) through \((4, 1)\) and the origin. Then, find a vector \(\vec{b}\) in \(\mathbb{R}^2\) such that the solution set of \(Ax = \vec{b}\) is not a line in \(\mathbb{R}^2\) parallel to the solution set of \(Ax = \vec{0}\). Why does it not contradict the theorem about the relation between the solutions of non-homogeneous and homogeneous with the same coefficient matrix.

Solution. For example, \(A = \begin{pmatrix}1 & -4 \\ 0 & 0\end{pmatrix}\). If \(\vec{b} = \begin{pmatrix}0 \\ 1\end{pmatrix}\), then there are no solutions, i.e., it is not a line. Also, it does not contradict the theorem as there are no solutions.

7. Consider the matrix

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

Find all solutions to \(Ax = 0\) and all solutions to \(Ax = \vec{b}\) given that

\[
\vec{p} = \begin{pmatrix}1 \\ 1 \\ 1\end{pmatrix}
\]

is a solution to the second equation. What is \(\vec{b}\)?

Solution. First we solve \(Ax = 0\) by row reduction.

\[
\begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
so the solutions to the homogeneous equation are

\[ x = t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \]

for \( t \in \mathbb{R} \).

Since we are given that \( p \) is a solution to \( Ax = b \). Theorem in class implies that solutions to this equation are of the form

\[ x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \]

for \( t \in \mathbb{R} \). We also then compute

\[ b = Ap = \begin{pmatrix} 1 + 1 + 1 \\ 1 + 0 + 1 \\ 0 + 1 + 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}. \]

8. If \( B = \begin{bmatrix} -1 & 3 & 5 \\ 3 & 8 & 2 \\ 2 & 5 & 1 \end{bmatrix} \), then the solution set of \( Bx = 0 \) is

(a) a single point; (b) a line in \( \mathbb{R}^3 \);

(c) a plane in \( \mathbb{R}^3 \); (d) the set of all points in \( \mathbb{R}^3 \).

Explain your answer.

\textit{Solution.} Bring the corresponding augmented matrix into row echelon form.

\[ \begin{pmatrix} -1 & 3 & 5 & 0 \\ 3 & 8 & 2 & 0 \\ 2 & 5 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 3 & 5 & 0 \\ 0 & 17 & 17 & 0 \\ 0 & 11 & 11 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 3 & 5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

Notice that \( x_1, x_2 \) are the basic variables and \( x_3 \) is the free variable. Using the parametric form of the solution, we will see that the solution has form a constant vector times \( x_3 \). Therefore, the solution is a line through \( 0 \) in \( \mathbb{R}^3 \). Therefore, the answer is (b). \( \square \)