(1)

The inverse of a matrix

number of rows = number of columns

Def. An $n \times n$ matrix A is invertible if there is an $n \times n$ matrix A^{-1} (notation) called inverse of A so that

$$A \cdot (A^{-1}) = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = A^{-1} \cdot A$$

Similar to: $5 \cdot \frac{1}{5} = 1 = \frac{1}{5} \cdot 5$

Difference: matrices do not commute in general

Note: 1. If A is invertible, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$

2. Inverse does not always exist

For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ doesn't have inverse $AB = BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

as for any $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{R}$

Important! A is invertible $\Longleftrightarrow$ if we view A as a map $\bar{x} \mapsto A\bar{x}$, then A is onto & one-to-one

$\Longleftrightarrow$ pivots in every row and column.

Thm If A is invertible $n \times n$ matrix, then for each $\bar{b}$ in $\mathbb{R}^n$, the equation $A\bar{x} = \bar{b}$ has the unique solution $\bar{x} = A^{-1}\bar{b}$

(2) How do we find inverse?

Row reduction gives an algorithm!

Example Find inverse of A:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

1. Set matrix with A on left and Identity on right of the same size as A.

$$[A \,\vdots\, I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

2. Bring the part A of matrix $[A \,\vdots\, I_3]$ to I_3 by row operations (as before just bring A to RREF). The inverse of A will be on the part where I_3 was originally ~~[illegible]~~

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \to \frac{R_3}{2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

③ Note! If you cannot bring A to Identity matrix of the same size, then A is not invertible!

Properties

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$(AB)^T = B^T A^T$

~~Properties~~ **Properties**

$(AB)^{-1} = B^{-1}A^{-1}$ if A, B are invertible

$(A^T)^{-1} = (A^{-1})^T$

Thm Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then $A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

If $ad - bc = 0$, then A is not invertible.

Def. If A is 2×2 matrix, then

$\det(A) = ad - bc$ - determinant of A.

Why algorithm described above works?

1. Let $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What is $E_1 A$?

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

corresponds to $R_1 \leftrightarrow R_2$ in A.

2. Let $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. What is $E_2 E_1 A$?

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

corresponds to $R_2 \to R_2 - R_1$ in $E_1 A$.

3. Let $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. What is $E_3(E_2 E_1 A)$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

corresponds to $R_3 \to \frac{R_3}{2}$ in $E_2 E_1 A$. !

4. Let $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. What is $E_4(E_3 E_2 E_1 A)$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I_3}$$

corresponds to $R_2 \to R_2 + R_3$ in $E_3 E_2 E_1 A$.

We got $\underline{E_4 E_3 E_2 E_1 A = I_3}$

E_1, E_2, E_3, E_4 are called elementary matrices.

Elementary matrices correspond to row operations.

<u>Note!</u> $A^{-1} = E_4 E_3 E_2 E_1$

(5) The invertible Matrix Theorem

A – $n \times n$ matrix.

The following statements are equivalent, i.e, either all true or all false.

1) A is an invertible matrix
2) A is row equivalent to the $n \times n$ indentity matrix I_n
3) A has n pivot positions
4) The equation $A\bar{x} = \bar{0}$ has only the trivial solution
5) The columns of A form a linearly independent set.
6) The linear transform. $\bar{x} \mapsto A\bar{x}$ is one-to-one.
7) The equation $A\bar{x} = \bar{b}$ has at least one solution for each $\bar{b}$ in $\mathbb{R}^n$
8) The columns of A span $\mathbb{R}^n$
9) The linear tr. $\bar{x} \mapsto A\bar{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
10) A^T is an invertible matrix

(6) Determinants ($\det(A)$)

1) A is 2×2 matrix; If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $\det(A) = ad - bc$

$\det(A) \neq 0 \longleftrightarrow A$ is invertible

2) A is 3×3 matrix: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

then $\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$

$\det(A) \neq 0 \longleftrightarrow A$ is invertible

3) ~~Want~~ to generalize to $n\times n$ matrices.

Important property! $\det(A) \neq 0 \longleftrightarrow A$ is invertible

Geometric interpretation.

Let A be $[2\times 2]$ **Thm** $|\det(A)|$ = Area of parallelogram spanned by $\bar{a}_1$ and $\bar{a}_2$

absolute value

where $A = \begin{bmatrix} - \bar{a}_1 - \\ - \bar{a}_2 - \end{bmatrix}$