(2) start from here

What is special about map $\mathbb{R}^n \to \mathbb{R}^m$ given by a matrix A?

Properties of matrix-vector product!

$A(\bar{u}+\bar{v}) = A\bar{u} + A\bar{v}$, where $\bar{u},\bar{v} \in \mathbb{R}^n$

$A(d\bar{u}) = d(A\bar{u})$, $d \in \mathbb{R}$ — scalar.

<u>Def</u> A <u>linear transformation</u> $T: \mathbb{R}^n \to \mathbb{R}^m$ is any map that satisfies both properties.

1) $T(\bar{u}+\bar{v}) = T\bar{u} + T\bar{v}$, where $\bar{u},\bar{v} \in \mathbb{R}^n$

2) $T(d\bar{u}) = d\,T(\bar{u})$, $d \in \mathbb{R}$

<u>Notice!</u> What is $T(\bar{0})$?

$\underline{T(\bar{0})} = T(\bar{0}+\bar{0}) = \underline{T(\bar{0}) + T(\bar{0})}$

$T(\bar{0}) = 2T(\bar{0}) \implies T(\bar{0}) = \bar{0}$

<u>Important!</u>

Any linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is given by some $m\times n$ matrix A!

Let $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ... $\bar{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.

Any vector in $\mathbb{R}^n$ $\bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\bar{e}_1 + v_2\bar{e}_2 + \dots + v_n\bar{e}_n$

If we know $T(\bar{e}_i)$, then we know $T(\bar{v}) = v_1 T(\bar{e}_1) + v_2 T(\bar{e}_2) + \dots + v_n T(\bar{e}_n)$

Matrix A of T is: $A = \begin{bmatrix} | & | & & | \\ T(\bar{e}_1) & T(\bar{e}_2) & \dots & T(\bar{e}_n) \\ | & | & & | \end{bmatrix}$

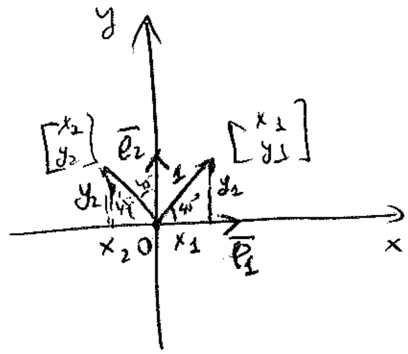
(3)

Example Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle 45°, with counterclockwise rotation for a positive angle. Find the matrix A of this transformation.

Solution:

$$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Need to find $T(\bar{e}_1)$ and $T(\bar{e}_2)$.



$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{x_1}{1} \Rightarrow x_1 = \frac{1}{\sqrt{2}}$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{y_1}{1} \Rightarrow y_1 = \frac{1}{\sqrt{2}}$$

$$y_2 = \frac{1}{\sqrt{2}}, \quad x_2 = -\frac{1}{\sqrt{2}}$$

$$T(\bar{e}_1) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad T(\bar{e}_2) = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A = \begin{pmatrix} | & | \\ T(\bar{e}_1) & T(\bar{e}_2) \\ | & | \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Example Fix $a, b, c \in \mathbb{R}$

Consider a map $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} c^3 x_3 \\ x_1 + b^2 x_2 \\ x_2 + a x_1 \end{pmatrix}$$

(1) Is T a linear transformation?

(i) $T(\bar{u} + \bar{v}) = T\left(\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}\right) = \begin{pmatrix} c^3(u_3 + v_3) \\ u_1 + v_1 + b^2(u_2 + v_2) \\ u_2 + v_2 + a(u_1 + v_1) \end{pmatrix} = \begin{pmatrix} c^3 u_3 \\ u_1 + b^2 u_2 \\ u_2 + a u_1 \end{pmatrix} + \begin{pmatrix} c^3 v_3 \\ v_1 + b^2 v_2 \\ v_2 + a v_1 \end{pmatrix} =$

Let $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$= T(\bar{u}) + T(\bar{v})$

(4)

2) $T(d\bar{u}) = T\left(\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}\right) = \begin{pmatrix} c^3 du_3 \\ du_1 + b^2 du_2 \\ du_2 + a du_1 \end{pmatrix} = d \begin{pmatrix} c^3 u_3 \\ u_1 + b^2 u_2 \\ u_2 + a u_1 \end{pmatrix} = d\,T(\bar{u})$

T is linear transformation.

(2) Find the matrix A of T?

$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\bar{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\bar{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$T(\bar{e}_1) = \begin{pmatrix} 0 \\ 1 \\ a \end{pmatrix}$, $T(\bar{e}_2) = \begin{pmatrix} 0 \\ b^2 \\ 1 \end{pmatrix}$, and $T(\bar{e}_3) = \begin{pmatrix} c^3 \\ 0 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 0 & c^3 \\ 1 & b^2 & 0 \\ a & 1 & 0 \end{pmatrix}$$

(3) For what values of a, b, c is T one-to-one? onto?

one-to-one: pivot in every column of A

onto: pivot in every row of A

Bring A into REF:

$$\begin{pmatrix} 0 & 0 & c^3 \\ 1 & b^2 & 0 \\ a & 1 & 0 \end{pmatrix} \xrightarrow[\text{after } R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & b^2 & 0 \\ a & 1 & 0 \\ 0 & 0 & c^3 \end{pmatrix} \rightarrow \begin{pmatrix} \boxed{1} & b^2 & 0 \\ 0 & 1-ab^2 & 0 \\ 0 & 0 & c^3 \end{pmatrix}$$

If $1 - ab^2 = 0$, then $\begin{pmatrix} 1 & b^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c^3 \end{pmatrix} \Rightarrow$ no pivot in 2nd column $\Rightarrow$ not one-to-one; no pivot in 2nd row $\Rightarrow$ not onto.

If $1 - ab^2 \neq 0$:

- $c^3 = 0$ $(c = 0)$ $\rightarrow$ no pivot in 3rd column and row $\Rightarrow$ not one-to-one, not onto
- $c^3 \neq 0$ $(c \neq 0)$ $\rightarrow$ pivot in every row and every column $\rightarrow$ one-to-one & onto

5. Is the following transformation linear?

a) $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix}$?

Consider $T\left(d\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = T\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} d^2 x_1 x_2 \\ d x_2 \end{pmatrix}$

$d\,T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = d\begin{pmatrix} x_1 x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} d x_1 x_2 \\ d x_2 \end{pmatrix} \neq$ if $d \neq 0$ $\Rightarrow$ not linear.

b) $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \cancel{\begin{pmatrix} \sqrt[3]{x_1^3 + x_2^3} \\ 0 \end{pmatrix}} \begin{pmatrix} (x_1^3 + x_2^3)^{1/3} \\ 0 \end{pmatrix}$

Consider $T\left(d\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} (d^3 x_1^3 + d^3 x_2^3)^{1/3} \\ 0 \end{pmatrix} = \begin{pmatrix} d\,(x_1^3 + x_2^3)^{1/3} \\ 0 \end{pmatrix} = d\begin{pmatrix} (x_1^3 + x_2^3)^{1/3} \\ 0 \end{pmatrix} =$

$= d\,T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$T\left(\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}\right) = \begin{pmatrix} \left((u_1+v_1)^3 + (u_2+v_2)^3\right)^{1/3} \\ 0 \end{pmatrix} \neq T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) + T\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$

$T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} (1+1)^{1/3} \\ 0 \end{pmatrix} = \begin{pmatrix} 2^{1/3} \\ 0 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 0 \end{pmatrix} \left(= T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\right)$