6 What can we do with matrices?

1. Add $m \times n$ matrices (matrices of the same size)

Entry of i-th row and j-th column of $A+B$: $(A+B)_{ij} = \underset{\substack{\uparrow \\ \text{of } i\text{-th row and } j\text{th column of } A}}{A_{ij}} + \underset{\substack{\uparrow \\ \text{of } i\text{-th row and } j\text{-th column of } B}}{B_{ij}}$

Example

$$\begin{pmatrix} 1 & 3 \\ -2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 7 & 0 \end{pmatrix} = \begin{pmatrix} 1+(-1) & 3+0 \\ -2+0 & 0+0 \\ 0+7 & 2+0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -2 & 0 \\ 7 & 2 \end{pmatrix}$$

Note! Cannot sum

$$\begin{pmatrix} 1 & 3 \\ -2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$ not defined as different size

2. Scale $m \times n$ matrix by number

$(cA)_{ij} = c\,A_{ij}$, where $c \in \mathbb{R}$ scalar

Example $-2 \cdot \begin{bmatrix} 3 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2\cdot 3 & -2\cdot 1 & -2\cdot 0 \\ -2\cdot 4 & -2\cdot 0 & -2\cdot 2 \end{bmatrix} = \begin{bmatrix} -6 & -2 & 0 \\ -8 & 0 & -4 \end{bmatrix}$

3. Multiply $m \times n$ matrix with $n \times p$ matrix

$$\underset{\substack{\uparrow \\ m\times n \text{ matrix} \\ (m \text{ rows}, n \text{ columns})}}{A} \cdot \underset{\substack{\uparrow \\ n\times p \text{ matrix} \\ (n \text{ rows}, p \text{ columns})}}{B} = \underbrace{\begin{array}{ccc} \text{1st column} & \text{2nd column} & p\text{-th column} \\ \left[\begin{matrix} | \\ (A\vec{b}_1) \\ | \end{matrix}\right. & \begin{matrix} | \\ (A\vec{b}_2) \\ | \end{matrix} \cdots & \left.\begin{matrix} | \\ (A\vec{b}_p) \\ | \end{matrix}\right] \end{array}}_{\substack{m\times p \text{ matrix} \\ (m \text{ rows}, p \text{ columns})}} \quad \text{if } B = \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \\ | & | & & | \end{bmatrix}$$

⑦ Alternative formula

If $A = \begin{bmatrix} \text{---} & \bar{a}_1 & \text{---} \\ \text{---} & \bar{a}_2 & \text{---} \\ & \vdots & \\ \text{---} & a_m & \text{---} \end{bmatrix}$ ← row vectors

$B = \begin{bmatrix} | & | & & | \\ \bar{b}_1 & \bar{b}_2 & \cdots & \bar{b}_p \\ | & | & & | \end{bmatrix}$ ← column vectors

$$(A \cdot B)_{ij} = \underset{\substack{i\text{-th} \\ \text{row} \\ \text{as vector}}}{\bar{a}_i} \underset{\text{dot product}}{\bullet} \underset{\substack{j\text{-th} \\ \text{column} \\ \text{as vector}}}{\bar{b}_j}$$

entry of i-th row and j-th column

<u>Example</u>

$$\underset{2\times 2}{\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}} \underset{2\times 3}{\begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}} = \begin{bmatrix} 2\cdot(-1)+(-3)\cdot 0 & 2\cdot(-1)+(-3)\cdot 1 & 2\cdot 0+(-3)\cdot 3 \\ 1\cdot(-1)+0\cdot 0 & 1\cdot(-1)+0\cdot 1 & 1\cdot 0+0\cdot 3 \end{bmatrix} =$$

$$= \begin{bmatrix} -2 & -5 & -9 \\ -1 & -1 & 0 \end{bmatrix}$$

Cartoon of AB as <u>composition</u> of first B, then A

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "B"] \arrow[rr, bend right, "A\cdot B"'] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

<u>Lemma</u>: $A(B+C) = AB + AC$

$(B+C)\cdot A = BA + CA$

8 <u>Important!</u>

1) $A \cdot B \neq B \cdot A$ in general!

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\not\parallel$$

$$B \cdot A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We can switch order only if $\underline{\underline{m = p}}$

2) $A \cdot B = \underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}_{\text{all entries } 0}$ $\overset{\text{doesn't imply}}{\not\Longrightarrow}$ $A = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$ or $B = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3) If $AB = AC$ and $A \neq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$, <u>not true</u> in general that $B = C$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \qquad \text{but } B \neq C$$

$$A \cdot C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(9) Special matrices

1) $$\mathbf{0} = \begin{array}{c} \\ m\text{-rows} \\ \\ \end{array} \underbrace{\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{pmatrix}}_{n\text{-columns}}$$ zero matrix

$$\boxed{A + 0 = A}$$

2) $$I_n = \begin{array}{c} \\ n\text{-rows} \\ \\ \end{array} \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}}_{n\text{-columns}}$$ identity matrix (1 only on diagonal)

$$\boxed{A \bullet I = A}$$

More operations for matrices

1) $$A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k\text{-times}}$$ where A - $n \times n$ matrix, k - positive integer

2) The transpose of a Matrix

A - $m \times n$ matrix

A^T = transpose of A — $n \times m$ matrix

$$(A^T)_{ij} = A_{ji}$$

entry of i-th row and j-th column

entry of j-th row and i-th column.

(10)

Example

$$A = \begin{pmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}$$

$$A^T = \begin{pmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Lemma

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $\boxed{(AB)^T = B^T A^T}$