(6) Determinants (det(A))

1) A is 2x2 matrix ; If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $\det(A) = ad - bc$

$\det(A) \neq 0 \iff A$ is invertible

2) A is 3x3 matrix : If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

then $\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} - a_{13}a_{22}a_{31} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$

$\det(A) \neq 0 \iff A$ is invertible

3) Want to generalize to nxn matrices.

Important property! $\det(A) \neq 0 \iff A$ is invertible

Geometric interpretation:

Let A be [2x2] **Thm** $|\det(A)|$ = Area of parallelogram spanned by $\bar{a}_1$ and $\bar{a}_2$

absolute value

where $A = \begin{bmatrix} - \bar{a}_1 - \\ - \bar{a}_2 - \end{bmatrix}$

(7) How both sides change under row operations

Row operation	det(A)	Area
Scale row and add to another	unchanged	unchanged (geometrically, shear)
interchange rows	det $\rightsquigarrow$ $-$det (determinant changes sign)	unchanged (geometrically, flip)
scale row by $\lambda \neq 0$	det $\rightsquigarrow$ λdet (determinant is multiplied by λ)	Area $\rightsquigarrow$ $\lvert\lambda\rvert$ Area (geometrically, scale)

want to be true for det of $n \times n$ matrix

<u>Inductive definition of det(A) for $n \times n$ matrices</u>

Suppose we already know determinant for $(n-1) \times (n-1)$ or smaller matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

(8) Fix i-th row, j-th column → cross them out of A ⇝ get $(n-1)\times(n-1)$ matrix A_{ij}

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

Set $M_{ij} = \det(A_{i,j})$ ← it is called (i,j)-minor

$C_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} \det(A_{ij})$ ← it is called (i,j)-cofactor

<u>Def.</u> $\det(A) = a_{11}C_{11} + a_{12}C_{12} + \ldots + a_{1n}C_{1n}$

We obtain cofactor expansion along the first row.

In terms of M_{ij}:

$$\det(A) = a_{11}M_{11} - a_{12}M_{12} + \ldots + (-1)^{1+n} a_{1n} M_{1n}$$

<u>Notation</u> $M = (M_{ij})$ ← (i,j)-entry is M_{ij} – matrix of minors

$C = (C_{ij})$ ← (i,j)-entry is C_{ij} – matrix of cofactors

Example

a) Compute M and C matrices

b) Compute determinant of A using cofactor expansion.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Solution 1. Compute M:

$$M = \begin{bmatrix} 6 & 0 & 3 \\ 4 & 3 & 2 \\ -3 & 0 & 3 \end{bmatrix}$$

← only needed for cofactor expansion along the first row.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 0 & 3 \\ -4 & 3 & -2 \\ -3 & 0 & 3 \end{bmatrix}$$

$$\det(A) = 1 \cdot 6 + 2 \cdot 0 + 1 \cdot 3 = \underline{\underline{9}}$$

Can we use other row for cofactor expansion?

Yes.
$$\boxed{\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}}$$

Easier computation if use 2nd row for given matrix A in example

$$\det(A) = \underset{\overset{\parallel}{0}}{a_{21}}C_{21} + a_{22}C_{22} + \underset{\overset{\parallel}{0}}{a_{23}}C_{23} = 3 \cdot 3 = \underline{\underline{9}}$$

↑ need to know only C_{22}

(10) Can we use column for cofactor expansion?

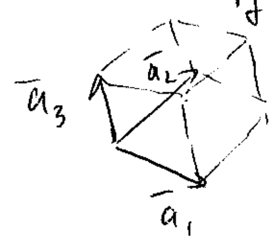
Yes.

$$\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}$$

Geometric interpretation

$|\det(A)|$ = Volume of parallelepiped constructed on $\bar{a}_1, \dots, \bar{a}_n$

if $A = \begin{bmatrix} — & \bar{a}_1 & — \\ — & \bar{a}_2 & — \\ & \vdots & \\ — & \bar{a}_n & — \end{bmatrix}$



Formula for the inverse!

Suppose A is invertible non matrix

Then $A^{-1} = \dfrac{1}{\det(A)} \cdot C^T$

(Notice! $\det(A) \neq 0$ if A is invertible)

For previous example $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

we have $A^{-1} = \dfrac{1}{9} \begin{bmatrix} 6 & -4 & -3 \\ 0 & 3 & 0 \\ 3 & -2 & 3 \end{bmatrix}$